Math 217 Daily Update Fall 2015 Section 5

This Daily Update will be written as soon as possible after I finish teaching at 11:30. It will summarize what we did, correct errors and make clarifications, provide important announcements, such as information about upcoming quizzes, and occasionally offer some general advice. Make a habit of checking it frequently.

Monday Dec 14: We took the FINAL QUIZ and then did some reviewing. I will hold office hours in our room the usual time Wednesday, and continue with my regular office hours after that. Study hard! The class webpage has some writeups that might be useful: 1) Change of Basis and all that 2) List of Definitions with examples and 3) Only the Definitions and Propositions. Good luck! Study hard!

Friday Dec 11: We worked on understanding the spectral theorem. I posted the worksheet WITH ANSWERS. This is slightly edited from the class worksheet so the numbers might be different. Mondays quiz will come DIRECTLY from the worksheet. I will take one problem from among problems 1, 2, 3 and 4, and one from 5, 6, 7 using the numbering on the online worksheet. EXTRA CREDIT FOR ANYONE WHO READS MY "LIST OF DEFINITIONS" and provides feedback. I will post that soon.

Wednesday Dec 9: We spent some time to connect the theoretical approach to "diagonalizing a transformation $T$" to "diagonalizing a matrix" as in the book. If $T : V \to V$ is a finite dimensional vector space, then we say $T$ is diagonalizable if there is a basis $B$ for $T$ so that the $B$-matrix of $T$ is diagonal. In this case, the basis $B$ will be an eigenbasis and the elements on the diagonal matrix will be the eigenvalues. (Make sure you see why this is true! If you don’t, please come to office hours so you can straighten this out before the exam!). In the special case where $V = \mathbb{R}^n$, then $T$ is given by left multiplication by some matrix $A$—this matrix $A$ can also be described as the matrix of $T$ in the standard basis. If we change coordinates to the eigenbasis $B$, then as usual with changing basis, we will have

$$[T]_B = S^{-1}[T]_S S$$

where $S$ is the standard basis and $S$ is the change of basis matrix from $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ to the standard basis. Note that $S$ is the $n \times n$ matrix

$$[\vec{v}_1 \quad \vec{v}_2 \quad \ldots \quad \vec{v}_n].$$

So this special case becomes Theorem 7.1.3 in the book:

$$S^{-1}AS = D$$

where $S$ is the matrix made up of the eigenbasis and $D$ is diagonal matrix with the corresponding eigenvalues on the diagonal. More details on the worksheet, which is posted.

Monday Dec 7: We took Quiz 11. We then went over it. We then practiced some proof problems. We showed that the (real) eigenvalues of an orthogonal transformation can be only $\pm 1$ (and that the complex eigenvalues also satisfy $|\lambda| = 1$). We showed that the characteristic
polynomials of similar matrices are the same. This is important because it means the characteristic polynomial of a transformation $T : V \to V$ is well-defined: we can compute it from any matrix representing $T$ in any basis! We have only one more week of class! The important things we still need to understand are complex eigenvalues and the idea of orthogonal diagonalization and the Spectral Theorem.

We then looked a little at what happens when some of the eigenvalues are not real numbers. Basically, all the same things hold. For example, we looked at the ”rotation through $\pi/2$ map” of $\mathbb{R}^2$. Its matrix in the standard basis is \[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}.
\] Its characteristic polynomial has roots $\pm i$. These are also eigenvalues! We can solve for complex eigenvectors, and see that each geometric multiplicity 1, so there are complex eigenvectors. So this matrix is diagonalizable over $\mathbb{C}$ but not $\mathbb{R}$.

Friday Dec 4: We postponed the quiz until Monday. Monday’s quiz will be taken (almost) exactly from the worksheet. We continued understanding eigenvectors, eigenvalues, eigenspaces, eigenbases, and diagonalization, as well as the algebraic and geometric multiplicity of eigenvalues. Please see the worksheet for Dec 4 to study for the quiz.

Wednesday Dec 2: We continued studying eigenvectors, eigenvalues, eigenbases and eigenspaces of a linear transformation $T : V \to V$. We also discussed the characteristic polynomial of $T$. Be sure you know the definitions for transformations, not just matrices. Note that the book defines these concepts “for a matrix.” When they say “eigenvector of matrix $A$” they mean “eigenvector of the linear transformation of $\mathbb{R}^n$ given by multiplication by $A$.” Given any finite dimensional vector space $V$ and linear transformation $T : V \to V$, the eigenvectors, eigenvalues, characteristic polynomial (etc) of $T$ are the same as the eigenvectors, eigenvalues, characteristic polynomial (etc) of any matrix representing $T$. You can chose ANY basis $B$ for $V$, find the matrix $[T]_B$ of $T$ in the basis $B$, compute the characteristic polynomial as $\det(A - xI)$. Its roots are the eigenvalues of $T$. It doesn’t matter what basis we use: they all have the same characteristic polynomial. We also discussed algebraic and geometric multiplicity. QUIZ FRIDAY. Be sure you know these definitions! Since not all groups got through all problems on the worksheet, the quiz will focus on definitions, examples, and the first page of the worksheet. Please finish the webwork early. It will help a lot with getting the most out of class.

Monday Nov 30: We discussed eigenvectors of a linear transformation $T : V \to V$. By definition, an eigenvector of $T$ is a vector $\vec{v} \in V$ such that $T(\vec{v}) = \lambda \vec{v}$ for some scalar $\lambda$. The scalar $\lambda$ is called the eigenvalue of the eigenvector $\vec{v}$. The amazing thing about eigenvectors is that if $V$ has a basis of eigenvectors, then the matrix of $T$ in this basis will be diagonal! This makes it easy to work with—for example, to compute larger powers of the matrix. Caution: Not every linear transformation has an eigenbasis! We did a worksheet in which students discovered a neat application of eigenvectors to computing fibonacci numbers. I have posted solutions to this worksheet. Make sure you can do all of it as you may see it again on a quiz.
Wednesday Nov 25: We worked on a worksheet to get students to really understand the geometric interpretation of the determinant. For a linear transformation defined by a \( n \times n \) matrix \( A \), we have \( |\det A| \) is the \( n \)-volume of the unit \( n \)-cube. Students proved this using QR factorization and remembering some dot product facts from Calc 3. Have a great Thanksgiving! I wish I could tell you to relax and forget about Math 217 for the weekend, but unfortunately, the problem sets go on, and someone thought it was a good idea to have Problem Set 9 (and webwork!) due Monday. Alas.

Monday Nov 23: We took Quiz 10. If you are not happy with how you did, you can turn in before 10:10 Wednesday for a better score. We then did some worksheet problems, including the fact that \( \det A = \det A^T \) for any (square) matrix, and that the determinant of an orthogonal matrix is either 1 or -1. [The remaining two worksheet problems take you through the proof that the absolute value of the determinant is the scale factor for area (in dimension two) or volume (in dimension 3). We did not get to these but will next time.] We then discussed the fact that the determinant is linear in each column/row and the fact that it is alternating, which means if we swap any two rows, the sign of the determinant switches. Finish webwork now! Don’t leave the problem set for Thanksgiving break!

Friday Nov 20: We learned how to compute the determinant of any square matrix by using a Laplace expansion along any row or column. We then did a worksheet in which the following secrets were revealed: A linear transformation \( \mathbb{R}^2 \to \mathbb{R}^2 \) takes a unit square to a parallelogram, whose area is \( |\det T| \) (note the absolute value!) Here, by determinant of \( T \) we mean the determinant of the matrix of \( T \) in the standard basis or any other basis. We proved that the determinant of similar matrices are equal so all matrices representing \( T \) in any basis have the same determinant. This works in any dimension: a linear transformation \( T \) of \( \mathbb{R}^3 \) will take a unit cube to a “parallelepiped” whose volume is \( |\det T| \). In higher dimensions, the analogous statement holds. Quiz Next Time: I will test that you can do problem F on the worksheet and one easy proof involving the determinant.

Wednesday Nov 18: Exam review.

Monday Nov 16: We took Quiz 9. We then went over it. This led naturally to a discussion of Fourier Analysis. The idea in Fourier analysis is to approximate an arbitrary continuous function \( h \) by linear combinations of trig functions: we want to find the “closest” trig-like function to \( h \). Remember that the “closest” element in some subspace can be found by projection onto the subspace. To talk about distances between elements and projections, we need an inner product.

To make this precise, we consider the space \( C^0[-\pi, \pi] \) of all continuous functions on the interval \([-\pi, \pi]\) is an inner product space with inner product

\[
\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} fg \, dx.
\]
The trig functions $f_n(x) = \sin(nx)$ and $g_m(x) = \cos(mx)$ (where $n, m$ can be any natural number) form a set of orthonormal vectors in this inner product space. (Check your understanding: what does this mean in terms of integrals?) They span a subspace of $C^0[-\pi, \pi]$. This gives us a way to approximate any continuous function as a linear combination of these trig functions. For example, consider the 34-dimensional space spanned by $f_n(x) = \sin(nx)$ for $n = 1, \ldots, 17$ and $g_m(x) = \cos(mx)$ for $m = 1, \ldots, 17$. [It is traditional to throw in the function constant function $\frac{1}{\sqrt{2}}$ as well, which is also orthonormal (think of it as the function $\cos(0x)$ if you like). The resulting space $T_{17}$ is 35 dimensional.] For any continuous function $h(x)$, we can easily compute the projection onto $T_{17}$: remember that in any inner product space, the projection onto a subspace $V$ is easy to compute if we are given an orthonormal basis $\{u_0, \ldots, u_{34}\}$ of $V$. Indeed:

$$ proj_V(h) = \langle h, u_0 \rangle u_0 + \langle h, u_1 \rangle u_1 + \cdots + \langle h, u_{34} \rangle u_{34} $$

So, in this case, the coefficient of $\sin(nx)$ in the projection of $h$ to $T_{17}$ is

$$ \langle h, \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} h \sin(nx) \, dx. $$

and likewise for cosine. These are called the Fourier coefficients of $h$. Recall that the importance of the projection to $T_{17}$ is that it is the closest function to $h(x)$ in $T_{17}$. So this gives us a way to write an arbitrary function as approximately linear combination of much more understandable functions (sines and cosines). If we instead project to $T_{100}$ or $T_{1000}$, we will get better and better approximations. This is very important in mathematical modelling, since arbitrary continuous functions arise in nature (think: average temperature in Ann Arbor as a function of minute over the last century) but can not be handled well by a computer. On the other hand, trig functions, which can be made to approximate our function very closely, are easy to handle. You should look at the book (Figure 9 on page 258) for a nice graphic of this idea. Please do both practice exams. We will go over them next time. The exam is Wednesday at 6 pm in East Hall 1324.

Friday Nov 13: We continued working in groups on inner product spaces. All students should now have had a chance to work on an inner product space involving continuous functions, and at least one other inner product space. The quiz next time will be based on the worksheets done wednesday and today. It will also have one question related to homework problem 3b that was discussed in class (symmetric matrices, transpose, etc). **EXAM II is next week, Wednesday!** It will cover everything up to Chapter 5 (including 5.5). Please start Homework Set 9, at least the problems on 5.5 (everything about inner products), because the exam will cover 5.5. There is reading webwork due monday on 6.1 but the determinant part of the homework (6.1, 6.2) can be focused on after the exam. The Section 5 website now has a write-up called “Change of Basis and all that” written by me with help from some students. Its purpose is to help you better understand some of the important points of the course that many students struggle with. **Anyone who turns in a marked-up copy of it by Monday 10:10**
am with good criticisms and/or a typo not found by another student will get some extra points on Monday’s quiz.

Wednesday Nov 11: We spent the whole class working through a worksheet to get you acquainted with inner product spaces. There were three different examples, each was worked out by two groups. We will continue next time, but your group will work on a different example. I also wrote up some Supplementary materials called “Change of Basis and All That.” Please download it and read. It should help with understanding crucial things for the exam.

Extra Credit people who give me typos or feedback on understandability.

Monday Nov 9: We took Quiz 8. If you are not satisfied with your performance, you can print out the version on line and turn it in BEFORE 10:10 am Wednesday. We then discussed 5.4. We found the vector in a given subspace $V$ of $\mathbb{R}^n$ closest to a given vector $\vec{x} \in \mathbb{R}^n$—it is the projection of $\vec{x}$ onto $V$. We can think of this as the vector $\vec{v} \in V$ which minimizes the distance $||\vec{x} - \vec{v}||$. We then talked about how to find the vectors closest to being solutions of some inconsistent linear system $A\vec{x} = \vec{b}$. These are called least squares solutions; be sure you understand and can state Definition 5.4.4. [Caution: the “least squares solutions” are not even usually solutions!] An important theorem is Theorem 5.4.5: a vector $\vec{x}^*$ is a least squares solution of $A\vec{x} = \vec{b}$ if and only if it is an actual solution of $A^T A \vec{x} = A^T \vec{b}$. That is, you can find the least squares solutions of $A\vec{x} = \vec{b}$ by multiplying both sizes by $A^T$ and then solving the resulting system, which (miraculously!) is always consistent. I said something wrong in class: the least squares solutions do not always form a vector space—they may not include the origin. However, just like solutions to any linear system, they are a translation of a subspace by one fixed vector (so look like, for example, a line or place in $\mathbb{R}^3$ that does not go through the origin). In particular: it is not true that the sum of two least squares solutions is a least squares solution in general! This will be true, however, if zero is a least squares solution.

Homework: Be sure to do the web homework due Wednesday, and get going on the problem set due Friday. Please re-read 5.4 and 5.5. These are hard sections!

Friday Nov 6: We worked on understanding what is an orthogonal transformation and orthogonal matrix. Be sure you understand the worksheet and T/F Quiz from today, especially the many different ways of thinking about orthogonal matrices presented in 5.3. We will have a QUIZ MONDAY which will include some of the exact same problems from the worksheet and/or QUIZ. We also went over an important application of $QR$ factorization for solving linear systems. We only have one more week before the exam, so please get going on PROBLEM SETS 8 A AND B. Of course, ideally, all webwork is done this weekend too.

Wednesday Nov 4: We talked about QR factorization and Gram-Schmidt. One important fact: let $B$ be any basis and let $A$ be the orthonormal basis obtained from $B$ via the Gram-Schmidt process. The change of basis matrix from any basis $B$ to $A$ is upper triangular. We then used this to understand the QR-factorization: let $M$ be the matrix whose columns are the elements of the basis $B$ (so $M$ is $d \times n$ if $B$ is a basis of a $d$-dimensional subspace of $\mathbb{R}^n$). Then
\[ M = QR \] where \( Q \) is the matrix whose columns are the elements of \( A \) and \( R \) is the change of basis matrix from \( B \) to \( A \). **Homework:** Webwork due Wednesday! Read 5.4 and do webwork before friday. Problem Set as usual. Less than 2 weeks until the exam!

**Monday Nov 2:** We took Quiz 7. If you are dissatisfied with your score and/or missed class, download it from the website, do it, and turn it in Wednesday for some points back. We then discussed the **Cauchy Schwarz Inequality** that says that for vectors in \( \mathbb{R}^n \), \[ |\vec{x} \cdot \vec{y}| \leq ||\vec{x}|| ||\vec{y}||. \] [Be sure you see why this makes sense, using the Math 215 definition of dot product as “product of magnitudes times cos of angle between.”] We then discussed **orthogonal transformations.** By definition, \( T : \mathbb{R}^n \to \mathbb{R}^n \) is orthogonal means that \( ||T(\vec{x})|| = ||\vec{x}|| \) for all vectors \( \vec{x} \in \mathbb{R}^n \). (You should understand that this means “\( T \) preserves lengths of vectors,” but be able to write down the precise definition in symbols). Although it is not obvious, orthogonal transformations also preserve the angle between vectors. That is, if the angle between two vectors \( \vec{x} \) and \( \vec{y} \) is \( \theta \), then after applying the orthogonal transformation \( T \), it is also true that the angle between \( T(\vec{x}) \) and \( T(\vec{y}) \) is also \( \theta \). We proved this by showing that orthogonal transformations preserves dot product: if \( T \) is orthogonal then \( T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y} \) for all vectors \( \vec{x}, \vec{y} \in \mathbb{R}^n \). **Homework: WEBWORK! Make good progress on your problem set!** We will be discussing 5.4 next time.

**Friday October 30:** We discussed the very important computational tool of **Gram-Schmidt Orthogonalization.** This is a technique for taking any basis of a subspace \( V \) of \( \mathbb{R}^n \) and transforming it into an orthonormal basis. Be sure you can do this technique. It will be on Monday’s quiz. Also, be sure you understand your mistakes on Quiz 6 and can do those problems on Quiz 7. We also proved the important theorem on the dimension of orthogonal complement: **if \( V \) is a subspace of \( \mathbb{R}^n \), then \( \dim V + \dim V^\perp = n \).** For homework: finish the webwork due Wednesday so we can make better use of classtime and get going on Problem Set 7. Of course, if you start early enough, you can ask me (and others) for help before/after class. You will need it on this one!

**Wednesday October 28:** We talked about 5.1, using a worksheet. The main ideas are **orthonormality.** Make sure you can precisely define an orthonormal set of vectors. Make sure you can prove any orthonormal set is linearly independent. Make sure you know an easy way to find the coordinates in an orthonormal basis. Make sure you know what is the **orthogonal complement** \( V^\perp \) of a subspace \( V \) of \( \mathbb{R}^n \). Make sure you know the **THEOREM** relating the dimensions of \( V \) and \( V^\perp \). Some students got to the proof of this theorem on the worksheet. **You should go through this and make sure you can prove it.** The proof is a cool application of the rank-nullity theorem. (You cannot get an A in Math 217 without being able to do these kinds of proofs. Many of you should be A students, even if you scored lower than this on Exam 1. It is not too late!!) **Homework:** Webwork 6 due tonight. Reading for 5.2 due by Friday, as is the problem set. This is the hardest one yet! Please take advantage of the "proof tutoring.”
Monday Oct 26: We took Quiz 6. We then discussed how to find the matrix $B = [T]_B$ of linear transformation $T : V \rightarrow V$ with respect to a given basis $B$ (Section 4.3), using the worksheet from last time. If $V$ has dimension $n$, the matrix will be $n \times n$. Its columns are the images of the elements in $B$, expressed again $B$. This is just like in Chapter 3 except that $V$ is no longer $\mathbb{R}^n$. By choosing a basis for $V$, we are in effect, fixing an isomorphism of $V$ and $\mathbb{R}^n$ (intuitively: we are fixing a way to think of the elements of $V$ as column vectors.) We also discussed the relationship between the matrices of the same linear transformation in different basis. Again, they are similar: $B = S^{-1}AS$ where $S$ is the “change of basis matrix from $B$ to $A$,” whose columns are the elements of $B$ expressed in $A$. [Note: the book’s words “change of basis matrix from $B$ to $A$ might be confusing;” be sure you know exactly what matrix to write down.]

Homework: Make sure you can do both sides of the Worksheet! Webwork 6 is due, also reading webwork on 5.1. Please make serious progress on Problem Set 6. It is the hardest yet!

Friday Oct 23: The point today was to understand linear transformations of vector spaces, §4.2. We did a worksheet, with half the class doing problem A and half problem B on a worksheet. Please read 4.3 carefully and do the reading webwork by Monday. Also, complete the web homework. The due date is wednesday but you will get much more out of it if you do it this weekend (all sections are covered already). ALSO: Problem Set 6 is on canvas. Get started now! I will announce the curve as soon as I get it (you may get it through canvas before I do).

Wednesday Oct 21: We went over problems from the two practice exams which were available on canvas.

Friday Oct 16: We discussed more about how to find the “change of basis” matrix $S$ witnessing the similarity of $A$ and $B$, when we know that $B$ and $A$ are two matrices of the SAME linear transformation. Let $T$ be a linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$. If $B$ is the matrix of $T$ with respect to basis $B$ and $A$ is the matrix of (the same!) $T$ with respect to basis $S$, then

$$B = S^{-1}AS$$

where $S$ is the matrix whose columns are the elements of basis $B$ expressed in $S$-coordinates. You should memorize this carefully. In the book (Theorem 3.4.4) this theorem is stated when $S$ is the standard basis, but when you state it carefully (as above), it works for any two bases.

We then discussed vector spaces (the book calls these linear spaces). There were six teams of students analyzing six different sets to see which are vector spaces.

Suki explained how her group showed that the set of all $2 \times 2$ matrices is a vector space. They found a basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. They also considered whether the subset of all invertible matrices is a subspace. It is NOT: the zero matrix is not invertible!

Stephen’s group showed that the solutions to the differential equation $y = y''$ is a vector space. It is a subspace of Siddharth’s group’s vector space, which was the set of all differentiable functions. Stephen and Jacob found that the two functions $f(x) = e^x$ and $g(x) = e^{-x}$
are elements of their vector space (because these functions equal their second derivative). They proved they are linearly independent (since the function \( f \) is not a scalar multiple of \( g \)). The \textbf{claimed} these functions are a basis, which is true but not obvious: how do we know there isn’t some other function equal to its second derivative that is not just a linear combination of these? (That is a Theorem in Math 316). They found an interesting subspace: the solutions to \( y = y' \) (which contains \( f(x) \) but not \( g(x) \), and in fact has \( f \) for a basis though again this is not obvious).

Kate’s group studied the set \( \mathcal{P} \) of all polynomials. They found that this set is a vector space with the usual ”Algebra I addition and scalar multiplication” of polynomials (combining like terms). The subset \( \mathcal{P}_2 \) of polynomials of degree two or less is a subspace, with basis \( \{1, x, x^2\} \). More generally, the vector space \( \mathcal{P} \) is not finite dimensional, since the polynomials \( 1, x, x^2, \ldots, x^n, \ldots \) are linearly independent. These infinitely many elements in fact also span \( \mathcal{P} \) so they are a basis, too.

The work done by the remaining two groups will be done next time.

\textbf{Homework:} Download BOTH practice exams from Canvas, but not the solutions (yet). Take both, without looking at the solutions. We will grade these in class Wednesday together.

\textbf{Wednesday Oct 14:} We continued discussing bases and coordinates, and especially finding the matrix of a linear transformation with respect to a non-standard basis (§3.4). Theorem 3.4.3 is essential! Make sure you can write down the matrix of a linear transformation with respect to a basis \( \mathcal{B} \) (remember its columns!). Theorem 3.4.4 is also important: it tells us how to relate the \( \mathcal{B} \)-matrix and the standard matrix \( A \) of a linear transformation. Basically, we have that the \( \mathcal{B} \)-matrix is \( S^{-1}AS \) for a particular invertible matrix \( S \) which Theorem 3.4.4 tells you how to find. You should memorize how to find the \( S \) in this theorem and make sure you know how to use it. For Friday: Read 4.1 and do the web-reading work. Finish the problem set! Remember: the FIRST MIDTERM EXAM IS NEXT WEDNESDAY, 6-8 pm. It will cover chapters 1, 2, and 3.

\textbf{Monday Oct 12:} We took quiz 5 (on webpage). We then discussed how to find the matrix of a linear transformation with respect to a given basis \( \mathcal{B} \), and what the relationship is between matrices of the same transformation with respect to different basis. We defined \textbf{similar} matrices (make sure you can too!). For next time: Web Homework 5 is due. Read 4.1 and do the webwork. Also, get going on Problem Set 5, which is the hardest yet and will be on the exam next week! Please plan to attend several professor’s office hours! My office hours have been filled with students from other sections....make sure the other professors’ are too!

\textbf{Friday Oct 9:} We discussed coordinates in bases that are not the standard basis, working on the whiteboard. We also did the B-side of a worksheet on finding the matrix of a linear transformation with respect to a basis other than the standard basis. Try to finish Webwork 5 this weekend. Quiz monday, on 3.2 and 3.3. Work on Problem Set 5!
Wednesday Oct 7: We discussed bases for subspaces, and dimension. Be sure you can state the definition from the book for basis, linearly independent, span, and dimension exactly. See the worksheet, activity on bases. Webwork!

Monday Oct 5: We reviewed finding bases for kernels and images of linear transformations. Note: the columns of the matrix of $T$ span the image so any linearly independent set of columns are a basis for the image. We talked about subspaces of $\mathbb{R}^n$. By definition, these are subsets closed under addition and scalar multiplication. Geometrically these are “linear spaces through the origin” (generalizations of lines, planes, etc). We practice scaffolding proofs. We took Quiz 4. See Worksheet and Quiz on the Section 5 webpage. Be sure you know the precise definitions of all words in boldface. The exam will have 5 definitions you will need to state. Homework: Webwork Homework, including Read 3.4. Be sure you get help on Problem Set 4, due Friday.

Friday Oct 2: We discussed/compute kernels and images of linear transformations. Be sure you understand the following points: The kernel is a linear space (through $\vec{0}$) in the domain. The image is a linear space in the target (through $\vec{0}$). Be sure you can find the kernel and image! If the transformation $T$ has matrix $A$, you can find the kernel by solving the system $A\vec{x} = \vec{0}$ (this is basically the definition, since the kernel is the set of $\vec{x}$ such that $T(\vec{x}) = \vec{0}$. You can find the image by finding the $\vec{y}$ such that the system $A\vec{x} = \vec{y}$ is INCONSISTENT. Both these processes involve row reduction.

We also talked about the span of a set of vectors, both algebraically and geometrically. If are not sure what this means, come to office hours! We also found bases for the kernel and image spaces. A Basis is a sets of vectors that span the kernel (or image) and is not redundant. We also took a "proof scaffolding quiz". We will do this every day until everyone can do it.

Please finish WEB HOMEWORK 4 this weekend! Do not wait for the last day....it will help you learn faster in class if you have already done it. Also, please start Problem Set 4! This way, you can ask for help on it in class and in office hours this week. I am happy to look at your proofs and tell you if they are correct.

Wednesday Sept 30: We took Quiz 3A, and then graded it (be sure you understand! Answers are posted on website). We discussed the IMAGE and KERNEL of linear transformations, using Worksheet 3.1. Please re-read 3.1 and be sure you understand image, kernel, injective, surjective. Webwork Homework due tonight! ALSO: Make serious progress on Problem Set 3, due Friday. It is hard, but worth it, to try to get a perfect score on part A and B, because the weekly problem sets (Part A and B) are a large part of your grade. Get into as many instructor's office hours as possible. All are listed on Canvas. FIRST MIDTERM EXAM IS THREE WEEKS FROM TODAY and will cover Chapters 1, 2, and 3 Monday.

Sept 28: We took Quiz 3. Some students scored very low—please figure out your errors and also please come see me! Quiz is posted, with answers. We then went over some T/F questions on the worksheet (this is page 3 on the worksheet posted for today). We then worked on the idea of invertible maps, and the words surjective, injective and bijective, using the worksheet.
A key idea: A linear transformation is invertible if and only if its matrix $A$ is an invertible matrix (a matrix $A$ is invertible if it has an inverse—that is, if there exists some matrix $B$ such that $AB = BA = I_n$.) For an $n \times n$ matrix $A$, this is the same as saying the rank of $A$ is $n$. **Homework:** Web homework due Wednesday! ALSO, start NOW on the problem set due friday. Please check canvas for instructor’s office hours that you can attend. Many of you need to get into office hours more! You should also read 3.2 and do the reading webwork for that.

**Friday Sept 25:** We saw two main new points from 2.3: 1) the composition of two linear transformations is always a linear transformation AND 2). the corresponding matrix of the composition is the PRODUCT of the corresponding matrices. That is, for the composition

$$S \circ T : \mathbb{R}^n \xrightarrow{T} \mathbb{R}^p \xrightarrow{S} \mathbb{R}^m,$$

we have

$$\text{matrix of } S \circ T = (\text{matrix of } S) \times (\text{matrix of } T).$$

**Make sure you understand this!** For example, check the sizes of the matrices work out. This was Problem C on todays worksheet, though we wrote it on the board (because I failed to print it for class). We also saw (Problem A on the worksheet) that projections onto a line are linear transformations (this required some Calc III many of you forgot!) and found the corresponding matrix. Problem B does the same for reflection. **QUIZ NEXT TIME:** I will literally take problems straight from the worksheet. Download the worksheet (with answers) from the Section 5 webpage. **Homework:** Read 3.1. Webwork. Also PLEASE get started on the problem set due friday and get yourself in any instructor’s office hours. All office hours are listed on the canvas site. Some of you had very low scores on the problem set: start earlier, work with others, and get help from any 217 instructor! This is a very hard class.

**Wednesday Sept 23:** We did quiz rewrites. Then we discussed proof techniques, specifically “induction” and also “proof by contradiction.” These are discussed in Appendix B, which has already been assigned— reread if necessary. The worksheet, with solutions, is posted on the webpage. We did not get to Problems A and B—those will be next time. We got bogged down because proofs are hard for so many students. **Answers to the worksheet are included on the webpage. Please look at it!** Proofs are hard—please get into my office hours, or any and every instructor’s office hours—all are listed in Canvas—to practice! **Homework:** Read 2.4 and do reading webwork on it. Also, web homework set 2 is due Wednesday. Weekly problem set is due Friday.

**Monday Sept 21:** We took Quiz 2. Then we worked on a worksheet to better understand linear transformations. A key point is this: every linear transformation can be described as matrix multiplication; today we focused on **how to find that matrix.** (Note: the definition of a linear transformation is not about matrices, despite the book’s naive definition.) We saw that for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we can write $T \vec{x}$ as $A\vec{x}$ where $A$ is the $m \times n$ matrix whose columns are the images $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, where the $\vec{e}_i$ are the standard unit vectors. **This is a crucial point:** Please come see me if you are unsure what this
MEANS. We then practiced writing down matrices for some common linear transformations, like rotations, simple reflections, etc. At the end, we started discussing compositions of transformations, which was a bit of a tangent...don’t sweat this, we will come back to it carefully later.

**Homework:** No new reading, but be sure you are caught up through 2.3 and appendix B, especially proof by induction. We will work on proof techniques next time. Web homework is due next time. Weekly problem set is due Friday.

Friday Sept 18: We took Quiz 1B to make sure you know the *Precise Definition* of a linear combination of vectors. We then discussed, using a worksheet (on webpage), the **very important** idea of a **linear transformation.** Roughly speaking, the Definition of a linear transformation between vector spaces is a mapping that respects vector addition and respects scalar multiplication. Please see Worksheet from 9/18 to be 100% sure you can correctly state the precise definition. (The definition in the book is not sophisticated enough for us, be sure you know the correct definition on the worksheet.) We saw some very important examples of linear transformations, which will be the main examples in 217, and even checked that they are equivalent. Namely: we saw that a map

\[
\mathbb{R}^n \rightarrow \mathbb{R}^m \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix},
\]

(where the \(a_{ij}\) are scalars) is a linear transformation— note here that the “coordinate entries” of the image vector are all linear expressions (no squares, no constant terms, no \(x_i x_j\), etc). We proved that this can be compactly written using matrix multiplication—namely,

\[
\mathbb{R}^n \rightarrow \mathbb{R}^m \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},
\]

where \(A\) is the \(m \times n\) matrix

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}.
\]

Be sure you see that these two are just two different ways of writing the same mapping from \(\mathbb{R}^n\) to \(\mathbb{R}^m\).

**Homework:** Read 2.3 and Appendix B. Webwork as usual—there is a lot again, note the “new problems” set due Wednesday has 36 problems! Find some people (from our section or any section) to work with on the weekly problem set due every friday. **Quiz Next time!** **Remember Monday quizzes count for the grade!**
Wednesday Sept 16: We took Quiz 1A, then graded it with tablemates. Be sure you understand it all now (on section website). We then talked about linear combinations using a worksheet. Be sure you can precisely state the definition of a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_d$. Be sure you understand the following theorem: a vector $\vec{b}$ in $\mathbb{R}^n$ is a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_d$ if and only if the system of linear equations $A\vec{x} = \vec{b}$ has a non-zero solution, where $A$ is the $n \times d$ matrix whose columns are the $\vec{v}_i$ (that is, $A = [\vec{v}_1 \vec{v}_2 \ldots \vec{v}_d]$). **Homework:** There is a ton of webwork due by 11:59. Also, more due Friday. Read 2.2. ALSO: THE FIRST PROBLEM SET WILL BE TURNED IN FRIDAY. TURN IN A AND B SEPARATELY. BE SURE TO DOWNLOAD THE CORRECT VERSION FROM CANVAS. No LATE PAPERS ACCEPTED. **Advice:** Some people are behind on webwork and not reading and keeping up. Please reconsider it Math 217 is right for you, and switch classes before the drop deadline!

Monday Sept 14: We took Quiz 1. If you missed it because of Rosh Hoshana, let me know. We discussed 1.3 by worksheet. Be sure you finish it. Answers to most of it are posted on the Section 5 webpage. Bonus points to the first student to find an error! We will begin with a quiz on it next time. Read 2.1. There is tons of webwork due Wednesday. Be sure to do it! Also, first problem set is due Friday. Make sure you acknowledge collaborators. **SORRY!** Due to a Doctor’s appointment, I am cancelling office hours this Wednesday. I will double them friday instead.

Friday Sept 11: Section 1.2. We worked on Worksheet 1.2 which covered Gauss-Jordan elimination and proof techniques. Be sure you precisely understand the following words: elementary row operation, row reduced echelon form (rref), pivot. Be sure you have practiced finding rref—HINT: it is best to focus on clearing out one column at a time, starting from the left. Be sure you understand the point of rref in solving systems of linear equations, and different ways to write out and interpret those solutions. Be sure you know how to see which variables are “free” in writing the solutions. We went through Worksheet 1.2, discussing A and C completely, with most students getting through B and D as well. Be sure you finish all of this at home! **Homework:** Read 1.3. Do relevant webwork. Reread Joy of Sets and Math Hygiene as needed. Finish worksheet. GET STARTED ON WEEKLY ASSIGNMENT, maybe with some classmates. Note: the “official” version of the homework will always be on CANVAS. I will also post the homework on the Section 5 website, but if changes are made to the official version I may not know about it, so make sure you are using the official (CANVAS) version to get full credit from the graders, who are working for all sections from the official version. The version on my website is for the convenience of people who can’t access canvas only. **QUIZ ON MONDAY** on material from 1.1, 1.2, including all handouts and worksheets.

Wednesday Sept 9: Material Covered from book: Section 1.1. We took Quiz 0 (link on my webpage). In going over the quiz, we reviewed and discussed the following ideas: the meaning of a “linear equation” and “a system of linear equations” both algebraically and geometrically. We saw, for example, that the solution set of one linear equation in three variables defines
a plane in $\mathbb{R}^3$, and that two linear equations in three variables usually defines a line in $\mathbb{R}^3$. You should be able to state clearly the two special cases when two linear equations in three variables fail to define a line, both algebraically and geometrically. You should also be able to explain analogously what to expect with three linear equations in three variables, and what kinds of exceptions might happen. All that Quiz 0 stuff should feel pretty familiar, from Math 215. If not, we can talk.

We then worked on Worksheet 1.1, the side dealing with proofs (problems B and C). We saw that figuring out the mathematics and figuring out the crystal clear logical step-by-step explanation from assumption to conclusion are two separate processes. Both are important in this class. You may need to struggle a lot with the second step, or both. This is normal. Just get help from everyone you can.

**Homework:** Read the two handouts, *Joy of Sets* and *Mathematical Hygiene*. Read Sections 1.1 and 1.2 in the book. Do the webwork (typically, each time there will be two parts, reading and exercises, but as of this writing, the whole system is down: keep checking!). Finish the worksheet: problems A and D remain. Homework One, parts A and B are due NEXT friday; you might want to take a look now.

**Advising:** If you are worried about how you did on Quiz 0, come talk to me or an advisor about whether Math 217 is the right class for you. If you don’t want to do proofs, you might prefer Math 417 (which is like Math 217 without the proofs) or Math 214 (ask an advisor). Also, Math 216 might be right for you. Keep in mind that Math 217 is a prerequisite for most higher math courses, and for the math major. Also, if you need a greater challenge, Math 420 or Math 295 (honors math, this is the hardest UM freshman math course) are options.