Math 217: §2.1 Linear Transformations
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Key Definition: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a map (i.e., a function) from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfying the following:

- $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ (that is, "$T$ respects addition").
- $T(a\vec{x}) = aT(\vec{x})$ for all $a \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ (that is, "$T$ respects scalar multiplication").

A. Suppose that $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ is a linear transformation. Suppose $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$.

1. Do we also know the value of $T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$? Find it, using only the definition of linear transformation given above. What about $T\begin{pmatrix} 2 \\ 0 \end{pmatrix}$? $T\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

**Solution note:** $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, $T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

2. Do we know the value of $T$ on any linear combination $a\vec{e}_1 + b\vec{e}_2$ where $\vec{e}_i$ are the standard unit column vectors in $\mathbb{R}^2$? Find it, using only the definition of linear transformation given above.

**Solution note:** Yes. $T(a\vec{e}_1 + b\vec{e}_2) = T(a\vec{e}_1) + T(b\vec{e}_2) = aT(\vec{e}_1) + bT(\vec{e}_2) = a\begin{pmatrix} 4 \\ 2 \end{pmatrix} + b\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4a - 2b \\ 2a - 2b \end{pmatrix}$.

3. What is $T\begin{pmatrix} a \\ b \end{pmatrix}$? Prove it.

**Solution note:** Done in (2) since $T\begin{pmatrix} a \\ b \end{pmatrix} = T(a\vec{e}_1 + b\vec{e}_2) = \begin{pmatrix} 4a - 2b \\ 2a - 2b \end{pmatrix}$.

4. Find a matrix $A$ such that $T\vec{x} = A\vec{x}$. Is it unique?

**Solution note:** $A = \begin{pmatrix} 4 & -2 \\ 2 & -2 \end{pmatrix}$. This is because $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4a - 2b \\ 2a - 2b \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$. It is unique, because its first column is determined by where $T$ sends $\vec{e}_1$ and its second column is determined by where $T$ sends $\vec{e}_2$. 
5. What does your matrix have to do with $T(\vec{e}_1)$ and $T(\vec{e}_2)$?

**Solution note:** The columns of $A$ are the column $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

B. Let $\vec{e}_1, \ldots, \vec{e}_n$ be the standard unit vectors for $\mathbb{R}^n$.

1. If we know the values of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^d$ on each $\vec{e}_i$, do we know the value for any $\vec{x} \in \mathbb{R}^n$? Why? Discuss with your tablemates.

2. Prove that $T(\vec{x}) = A\vec{x}$ where $A$ is the $d \times n$ matrix formed by the vectors $T(\vec{e}_1), \ldots, T(\vec{e}_n)$.

**Solution note:**

1). Yes. Every vector is a linear combination of the $\vec{e}_j$, so the image will be the corresponding linear combination of the $T(\vec{e}_j)$.

2). We restate this as a **Theorem:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $A$ be the $m \times n$ matrix whose $j$-th column is the $m \times 1$ vector $T(\vec{e}_j)$ for $\vec{e}_j$ the $n \times 1$ column vector which has all zeros except in the $j$-th spot, where there is a 1.

Then for all $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have

$$T(\vec{x}) = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$ 

That is, every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by a matrix multiplication by some $m \times n$ matrix $A$. The columns of $A$ are easy to find: they are the images under $T$ of the standard unit vectors in $\mathbb{R}^n$.

**Proof:** We can write $\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n$ for some scalars $x_i$. Using the definition of linear transformation, we have

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n) = T(x_1 \vec{e}_1) + T(x_2 \vec{e}_2) + \cdots + T(x_n \vec{e}_n) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n),$$

which is also the matrix product

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$ 

QED.

**This is a crucial idea. Be sure you understand exactly how a linear transformation can be described using matrix multiplication, and how to get the matrix.**
Linear transformations in geometry

C. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be dilation by a factor of three.
   1. Give a geometric reason that $S$ is a linear transformation using the definition.
   2. What is the associated matrix $A$ so that $S(\vec{v}) = A\vec{v}$?
   3. What about dilation (or contraction) by an arbitrary factor?

D. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation in the counter-clockwise direction by 90° (fixing the origin).
   1. Give a geometric explanation why $L$ is a linear transformation using the definition.
   2. What is the associated matrix $A$ so that $L(\vec{v}) = A\vec{v}$?
   3. What about rotation through an arbitrary angle $\theta$? To write the matrix, you need to remember your high school trig.

E. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the $x$-axis.
   1. Show that $M$ is linear by writing down a formula for it explicitly.
   2. What about reflection over the line $y = x$? Is this a linear transformation? If so, find its matrix.

F. Let $Q : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that stretches vertically by a factor of two and contracts horizontally by a factor of 3.
   1. Show that $Q$ is linear by writing down a formula for it explicitly.
   2. What about arbitrary (but different) scale factors vertically and horizontally? What happens if they are negative?

H. Bonus: Think geometrically: Do you think that reflection over an arbitrary line through the origin is a linear transformation? Can you write down its matrix?