Algebraic Closure. A field $K$ is algebraically closed if every polynomial $g(x) \in K[x]$ has a root.

1. Show that if $K$ is algebraically closed, then there is no irreducible polynomial of degree greater than one over $K$.
2. Show that any field $F$ is contained in a field $\bar{F}$ which is algebraically closed. [Hint: If $F$ is not algebraically closed, start by considering $F[x]/(g(x))$ where $g$ is irreducible over $F$.]
3. Show that if $F \subset K$, where $K$ is algebraically closed, there is a unique subfield $\bar{F}$ of $K$ containing $F$ which is also algebraically closed. Such a field is called an algebraic closure of $F$.
4. Show that the algebraic of $F$ is unique up to isomorphism fixing $F$. [Hint: start by showing that if $K$ and $K'$ are both algebraic closures, show that for each $\alpha \in K$, we can chose an element $\beta$ in $K'$ such that $F(\alpha) \cong F(\beta)$ by an isomorphism fixing $F$.] We denote any algebraic closure of $F$ by $\bar{F}$ and call it “the” algebraic closure of $F$.

2. Separable Extensions. An irreducible polynomial $f \in F[x]$ is separable if it has no multiple roots in any field extension. Let $f$ be an irreducible polynomial in $F[x]$.

1. Show $f$ is separable unless the derivative $f'$ is 0.
2. An algebraic field extension $F \subset K$ is separable if for every $\alpha \in K$, the minimal polynomial of $\alpha$ over $F$ is separable. Show that if $F$ has characteristic zero, then algebraic extension field of $F$ is separable over $F$.
3. Let $F = \mathbb{F}_p(t)$. Show that the polynomial $g(x) = x^p - t \in F[x]$ is irreducible but not separable.
4. With $g$ as in (3), compute the splitting field $K$ of $g$ over $F$. How many distinct roots does $g$ have? What is the degree of $K$ over $F$. Find its Galois group.
5. Find a necessary and sufficient condition (not involving the derivative) for a polynomial to be non-separable over $F$.

3. Normal Extensions. An extension $F \subset K$ of fields is normal if for any polynomial $g \in F[x]$ which has a root in $K$, all roots of $g$ are in $K$.

1. Show that a quadratic extension is always normal.
2. Which as the following extensions are normal: $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{7})$; $\mathbb{Q} \subset \mathbb{Q}(\eta_{11})$ where $\eta_{11}$ is a primitive 11-th root of unity; $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega$ is a primitive third root of unity.
3. Show that if $F \subset K$ is a splitting field for some polynomial $g \in F[x]$, then $F \subset K$ is normal.

4. Let $F$ be any field, and let $H$ be a finite subgroup of the group $F^\times$ of order $n$. Prove that $H$ is cyclic, and consists of the $n$-th roots of unity in $F$.

5. Let $f(x) \in F[x]$ be a polynomial of degree $n$, and let $K$ be a splitting field, which is to say, let $K = F(\alpha_1, \ldots, \alpha_n)$ where the $\alpha_i \in \bar{F}$ are roots of $F$ in an algebraic closure of $F$.

1. Show that $[K : F] \leq n$!
2. Show that if equality holds in (1), then the Galois group is isomorphic to $S_n$.

6. Show that the Galois group of $t^4 - 2$ over $\mathbb{Q}$ is isomorphic to the dihedral group $D_4$ of symmetries of a square.

Artin: 13.6: # 5, #, 7, #11; 13.8: #2; 14.1 #6, #10, #17, #18.