1. (18 points) Prove or disprove the following three statements, where $R$ is a commutative ring.

a). Let $0 \to A \to B \to C \to 0$ be an exact sequence of $R$-modules. If $A$ and $C$ are finitely generated, then $B$ is finitely generated.

**TRUE.** Let $a_1, \ldots, a_n$ be generators for $A$ and $c_1, \ldots, c_m$ be any lifts to $B$ of generators $c_1, \ldots, c_m$ of $C$ via the surjective map $B \to C$. Then $B$ is generated by $\{a_1, \ldots, a_n, c_1, \ldots, c_m\}$. Indeed, take an arbitrary element $b \in B$. Its image in $C$ can be written $\sum_{i=1}^m r_i c_i$. Then the element $b - \sum_{i=1}^m r_i c_i$ of $B$ is in the kernel of $B \to C$, which means it is some $R$-linear combination of the elements $a_1, \ldots, a_n$. So an arbitrary element of $B$ is in the $R$-span of $\{a_1, \ldots, a_n, c_1, \ldots, c_m\}$. QED.

b). If $M$ is a non-cyclic $R$-module, then $\wedge^2 M$ is non-zero.

**FALSE.** Consider the $\mathbb{Z}$-module $\mathbb{Q}$. It is not cyclic yet $\mathbb{Q} \wedge \mathbb{Q}$ is zero.

c). If $M$ is any module over a domain $R$, then the $R$-module $\text{Hom}_R(M, R)$ is torsion free.

**TRUE.** Say $\phi : M \to R$ is killed by some non-zero $r$. Then for all $m \in M$, we have $r\phi(m) = 0$. But $R$ is a domain, so this forces $\phi(m) = 0$ for all $m$, which means $\phi$ is the zero map. QED

2. (16 points) Find all prime ideals in the ring $\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}[x, y]/(x^3) \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x, y]/y^2)$.

We have isomorphisms

$$\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}[x, y]/(x^3) \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x, y]/y^2) \cong \mathbb{Z}/10[x, y]/(x^3, y^2) \cong \mathbb{Z}[x, y]/(10, x^3, y^2)$$

The first arrow comes from a general fact proven in class, the second uses the exact same idea (or you can apply a general "extension of scalars" fact proved in the homework). Any prime in this ring corresponds to a prime in $\mathbb{Z}[x, y]$ containing $(10, x^3, y^2)$. By definition of primeness, since such a prime must contain also $x$ and $y$, and either 2 or 5. So any prime must contain either $(x, y, 2)$ or $(x, y, 5)$. But both these ideals are maximal (as taking their quotients produces the fields $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ respectively). Hence these only two prime ideals of our ring are the images of these two in the quotient ring $\mathbb{Z}[x, y]/(10, x^3, y^2)$.

[If you want, you can write them as $(2 \otimes 1 \otimes 1, 1 \otimes x \otimes 1, 1 \otimes 1 \otimes y)$ and $(5 \otimes 1 \otimes 1, 1 \otimes x \otimes 1, 1 \otimes 1 \otimes y)$, but this is not really necessary.]
3. (16 points) Let $M$ and $L$ be modules over a commutative ring $R$, and let $\text{SYM}^k(M, L)$ be the set of $k$-multilinear symmetric maps over $R$

$$M \times M \times \ldots \times M \to L.$$ 

a). Describe a natural $R$-module structure on the set $\text{SYM}^k(M, L)$.

We add $\phi$ and $\psi$ by adding their values in $L$, and also multiply by elements of $R$ by multiplying the values in $L$ by elements of $R$. These maps are still obviously symmetric, and the $R$-module structure on $L$ obviously induces a $R$-module structure on the set $\text{SYM}(S^kM, L)$.

b). Show that there is a $R$-module isomorphism $\text{SYM}^k(M, L) \cong \text{Hom}_R(S^k M, L)$.

Given an element of $\phi$ of $\text{SYM}^k(M, L)$, we use the universal property of $S^k(M)$ to construct a unique $R$-module map $f(\phi)$ from $S^k(M)$ to $L$. We claim that the association $f$ is an $R$-module map: indeed, since $f(\phi + \psi)$ sends the class of the tensor $m_1 \otimes \ldots \otimes m_k$ to $\phi(m_1, \ldots, m_k) + \psi(m_1, \ldots, m_k)$, whereas $f(\phi)$ (respectively $f(\psi)$) sends the class of the tensor $m_1 \otimes \ldots \otimes m_k$ to $\phi(m_1, \ldots, m_k)$ (respectively $f(\psi)$), and such classes of tensors generate $S^k(M)$, we clearly have $f(\phi + \psi) = f(\phi) + f(\psi)$. Likewise, $f(r\phi)$ sends the class of the tensor $m_1 \otimes \ldots \otimes m_k$ to $r\phi(m_1, \ldots, m_k)$, which is clearly the same $rf(\phi)$ applied to the same class. Thus $f$ is an $R$-module homomorphism. Finally, $f$ is injective, since if $f(\phi) = 0$, then from the commutative diagram in the universal property it follows that $\phi$, which is the composition $M \times \ldots \times M \to S^k(M)f(\phi) \to L$ is also zero. It is also surjective: given any $\Phi \in \text{Hom}(S^k M, L)$, the composition $M \times \ldots \times M \to S^k(M)\Phi \to L$ is an element $\phi$ of $\text{SYM}^k(M, L)$, so by uniqueness of the map $f(\phi)$, we have $f(\phi) = \Phi$.

4. (16 points) Let $A$, $B$, and $C$ be finite dimensional vector spaces over a field $F$ of dimensions $a, b$ and $c$, respectively. Compute the dimension of

$$\left[ \text{Hom}_F(\bigwedge^k A, \bigwedge^\ell B) \otimes_F (T^m C)^* \right] \bigoplus_p \bigwedge^p (A \bigoplus B).$$

Since $\bigwedge^k A$ has dimension $\binom{a}{k}$, and $\bigwedge^\ell B$ has dimension $\binom{b+\ell-1}{b-1}$, clearly $\text{Hom}_F(\bigwedge^k A, \bigwedge^\ell B)$ has dimension $\binom{a}{k} \binom{b+\ell-1}{b-1}$. Since $(T^m C)$ has dimension $c^m$, so does its dual. Also, $\bigwedge^p(A \bigoplus B)$ has dimension $\binom{a+b}{p}$. Putting this all together, we get the dimension is

$$\binom{a}{k} \binom{b+\ell-1}{b-1} c^m + \binom{a+b}{p}.$$
5. (16 points) Let \( V \) and \( W \) be finite dimensional vector spaces over a field \( F \) of dimensions \( m \) and \( n \) respectively. Fixing bases \( \{v_1, v_2, \ldots, v_m\} \) for \( V \) and \( \{w_1, w_2, \ldots, w_n\} \) for \( W \), consider the map

\[
V \times W \to M_{m \times n}(F)
\]

\[
(v, w) \mapsto \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_m
\end{pmatrix} \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{pmatrix}
\]

where \( v = a_1 v_1 + a_2 v_2 + \ldots + a_m v_m \) and \( w = b_1 w_1 + b_2 w_2 + \ldots + b_n w_n \).

a). Show that this map induces an isomorphism of \( F \)-vector spaces, \( V \otimes W \cong M_{m \times n}(F) \).

The map is bilinear over \( F \) because matrix multiplication is bilinear. Thus it induces a \( F \)-vector space map \( T : V \otimes W \to M_{m \times n}(F) \). Note that this linear transformation sends the element \( v_i \otimes w_j \) to the matrix whose \( ij \)-entry is 1 and all other entries are zero. Thus \( T \) sends a basis of \( V \otimes W \) to a basis of \( M_{m \times n}(F) \), so must be an isomorphism.

b). Prove that under this isomorphism, the simple tensors \( v \otimes w \) are in one-to-one correspondence with the rank one \( m \times n \) matrices.

A simple tensor \( v \otimes w \) is sent to the \( n \times n \) matrix \( \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix} \), whose rows are clearly all scalar multiples of the row matrix \( \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix} \). Thus the rows are all linearly dependent, and the matrix is rank one. Conversely, given a rank one matrix, its rows are all scalar multiples of eachother. Some row, which without loss of generality we can assume to be the first, is non-zero. So labeling the rows \( R_1, R_2, \ldots, R_m \), there exist scalars \( a_2, \ldots, a_m \) such that \( a_i R_i = R_1 \) for all \( i \geq 2 \). If \( R_1 \) is the row matrix \( (b_1, \ldots, b_n) \), this exactly means that the matrix factors as \( \begin{pmatrix} 1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix} \), and so corresponds to the simple tensor \( \sum_{i=1}^m a_i v_i \otimes \sum_{j=1}^n b_j w_j \) under the transformation \( T \) (where \( a_1 = 1 \)).

c). Assuming \( n, m \geq 2 \), find an explicit element in \( V \otimes W \) that is not simple.

The tensor \( v_1 \otimes w_1 + v_2 \otimes w_2 \) is not simple, from parts a and b.

[Note that this exercise shows that, at least if \( m \) and \( n \) are greater than one and \( F = \mathbb{R} \), the simple tensors form a "set of measure zero" among all tensors.]
6. (16 points) Let $R$ be a commutative ring, and $I$ and ideal in $R$. Let $S$ be the subset of the polynomial ring $R[t]$ consisting of polynomials $f(t) = \sum a_i t^i$ where $a_i \in I$.

a). Prove that there is a surjective $R$-algebra map from the symmetric algebra $S(I)$ of the $R$-module $I$ onto $S$.

We note first that $S$ is a commutative $R$-algebra. Now define a map $I \rightarrow S$ sending $x \mapsto xt$. Since $(rx)t = r(xt)$ and $(x+y)t = xt + yt$, this is clearly an $R$-module map. By the universal property of the symmetric algebra, it extends to an $R$-algebra map $\Phi : S(I) \rightarrow S$. It remains only to show that this map is surjective. For this, note that $I^k$ is generated by products of the form $x_1 \ldots x_k$, where each $x_i \in I$. Thus $S$ is generated (as an $R$-module) by polynomials of the form $x_1 \ldots x_k t^k$, as we range over all values of $k \geq 0$. Such an element is the image, under $\Phi$, of the class of the simple tensor $x_1 \otimes \ldots \otimes x_k$ in $S^k(M) \subset S(M)$.

b). In the case where $R$ is a PID, prove that this map is an isomorphism.

If $I$ is the zero ideal, the result is trivial, as both rings are just $R$. Otherwise, $I = (a)$ for some $a \in R$, and hence is a free $R$-module of rank 1. Hence each $S^k(I)$ is a free $R$-module of rank 1 as well, with generator the class of $a \otimes a \otimes \ldots \otimes a$ ($k$ times), let us denote this generator by $\otimes^k a$. So an arbitrary element of $S(I)$ is of the form $\sum_{i=0}^k r_i (\otimes^i a)$, and under $\Phi$ this element maps to $\sum_{i=0}^k r_i a^{i}t^i$. So if such an element were sent to zero, each coefficient $r_i a^i$ must be zero. But $a^i$ is a non-zero element in a domain $R$, so this forces each $r_i = 0$. This shows the kernel of $\Phi$ is trivial, concluding the proof that $\Phi$ is an isomorphism.

EXTRA CREDIT: Give an example where this map is not an isomorphism.

I will still accept solutions for this.

[The ring $S$ is called the Rees ring of the ideal $I$, and plays an important role in the operation of "blowing up" in algebraic geometry.]