Math 156  Applied Honors Calculus II  Review Sheet for Final Exam  Fall 2016  
For full credit, justify your answer, and give the units if appropriate.  
1. True or false? Justify your answer with a reason or counterexample.  
a) \( \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n} = \frac{n+1}{2} \)

b) If \( \Delta x = \frac{b-a}{n} \) and \( x_i = a + i\Delta x \), then \( \lim_{n \to \infty} \sum_{i=1}^{n} f'(x_i)\Delta x = f(b) - f(a) \).

c) If the integral \( \int_{a}^{b} f(x) \, dx \) is approximated by the right-hand Riemann sum and the number of intervals \( n \) is doubled, then the error in the approximation decreases by a factor of \( \frac{1}{4} \).

d) If \( f(0) = f(1) = g(0) = g(1) = 0 \), then \( \int_{0}^{1} f(x)g''(x) \, dx = \int_{0}^{1} f''(x)g(x) \, dx \).

e) \( \int_{0}^{\infty} \frac{dx}{x^2} \) is a convergent improper integral.

f) A spring has natural length 20 cm. If 2 Joule of work is needed to stretch the spring from length 20 cm to 30 cm, then 4 Joule of work is needed to stretch it from length 30 cm to 40 cm.

g) The center of mass of the region \( \{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq \cosh x\} \) is \((\bar{x}, \bar{y}) = (0, \frac{1}{2})\).

h) If \( f(x) \) is the pdf of a random variable with mean \( \mu \), then \( f(x) \) has its maximum value at \( x = \mu \).

j) A certain radioactive material has a half-life of 100 years. If a given sample has mass 1 kg, then there will be 0.25 kg remaining after 400 years.

k) If $1000 is invested at 5% interest compounded continuously, then after 2 years the investment is worth between $1105 and $1112.

l) \( y(t) = 0 \) is a stable constant solution of the differential equation \( y' = y(1-y^2) \).

m) If a differential equation \( y' = f(y) \) has a constant solution \( y_1(t) = c, \) and \( y_2(t) \) is another solution with initial condition \( y_2(0) \neq c, \) then \( \lim_{t \to \infty} y_2(t) = c. \)

n) If a differential equation \( y' = f(y) \) is solved by Euler’s method, and the step size \( \Delta t \) decreases by a factor of \( \frac{1}{2}, \) then the error in the numerical solution increases by a factor of approximately \( \frac{1}{2}. \)

o) If \( \lim_{n \to \infty} a_n = 0 \) and \( \lim_{n \to \infty} b_n = \infty, \) then \( \lim_{n \to \infty} a_nb_n = 0. \)

p) If \( 0 \leq a_n \leq b_n \) and \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} b_n \) also converges.

q) \( \sum_{n=1}^{\infty} \frac{1}{n^2} < \int_{1}^{\infty} \frac{dx}{x^2} \)  

r) \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 0 \)

s) \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty \quad - \quad \infty = 0 \)

t) The alternating series test can be used to show that \( \sum_{n=0}^{\infty} (-1)^n \) diverges.

u) The ratio test can be used to show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

v) If the power series \( \sum_{n=0}^{\infty} c_nx^n \) converges for \( x = 1, \) then it also converges for \( x = -1. \)

w) If the power series \( \sum_{n=0}^{\infty} c_n(x-1)^n \) converges for \( x = 2, \) then it also converges for \( x = \frac{1}{2}. \)

x) \( \frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n(n+1)x^n \) for \( |x| < 1 \)  

y) If \( f(x) = e^{-x^2}, \) then \( f^{(6)}(0) = 0, \) \( f^{(6)}(0) = -6. \)

z) \( 2 < e < 3 \)  

aa) \( \int_{0}^{1} e^{-x^2} \, dx > \frac{2}{7} \)  

bb) \( \frac{\pi}{2} - \frac{1}{3!}(\frac{\pi}{2})^3 + \frac{1}{5!}(\frac{\pi}{2})^5 - \frac{1}{7!}(\frac{\pi}{2})^7 + \cdots = 1 \)

cc) \( \cosh^2 x - \sinh^2 x = 1 \)  

dd) \( \int \tan x \, dx = \sech^2 x \)

e) If \( T_1(x) \) is the first degree Taylor polynomial for \( f(x) \) at \( x = a, \) then the graphs of \( f(x) \) and \( T_1(x) \) have the same slope at \( x = a. \)
ff) If $f^{(n)}(0) = 0$ for all $n$, then $f(x) = 0$ for all $x$.

gg) $0.895 < e^{-0.1} < 0.905$  

hh) $\sqrt{1 + x^2} = 1 + x^2 + \cdots$  

ii) $\cosh ix = \cos x$

jj) $\int_0^{\pi/2} \sin^2 \theta d\theta = \int_0^{\pi/2} \cos^2 \theta d\theta$

kk) $\log(-1) = \pi i$

ll) $\binom{6}{3} = 2$  

mm) $(\frac{7}{4}) < (\frac{7}{3})$  

nn) $\sum_{n=0}^{k} \binom{k}{n} (1-n)^n = 0$

2. Evaluate the limit.

a) $1 + \frac{2016}{2017} + \left( \frac{2016}{2017} \right)^2 + \left( \frac{2016}{2017} \right)^3 + \cdots$

b) $\lim_{n \to \infty} \sum_{i=1}^{n} \left( 1 + \frac{i}{n} \right) \cdot \frac{1}{n}$

c) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i^n} \cdot \frac{1}{n}$

d) $\lim_{x \to 0} \frac{\sin x}{x}$

e) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

f) $\lim_{x \to \infty} (1 + \frac{x}{n})^{2n}$

g) $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$

h) $\lim_{x \to 0} \frac{(x+h)^4 - x^4}{h^4}$

i) $\lim_{h \to 0} \frac{(x+h)-f(x)}{h}$

j) $\lim_{h \to 0} \frac{(x+h)-2f(x)+f(x-h)}{h^2}$

k) $\lim_{h \to 0} \frac{1}{h} \int_0^h f(x) \, dx$

l) $\lim_{h \to 0} \frac{1}{h} \int_0^h x f(x) \, dx$

3. Find the antiderivative.

a) $\int e^{-x} \, dx$

b) $\int xe^{-x} \, dx$

c) $\int e^{-x^2} \, dx$

d) $\int xe^{-x^2} \, dx$

e) $\int x \sin x \, dx$

f) $\int e^{-x} \sin x \, dx$

g) $\int \frac{dx}{4x^2}$

h) $\int x \frac{dx}{4x^2}$

i) $\int \frac{dx}{\sqrt{4+x^2}}$

j) $\int \frac{dx}{\sqrt{1+x^2}}$

k) $\int \frac{dx}{4-x^2}$

l) $\int \frac{dx}{\sqrt{4x-x^2}}$

m) $\int \frac{dx}{\sqrt{4x-x^2}}$

n) $\int \sin^2 x \, dx$

o) $\int \sin^3 x \, dx$

p) $\int \sin^4 x \, dx$

4. Evaluate the integral.

a) $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} \, dx$

b) $\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$

c) $\int_{-\infty}^{\infty} (x-1)^2 \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} \, dx$

5. Show that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \, dx = \frac{\pi}{4}$. (hint: substitute $u = \frac{\pi}{2} - x$)

6. Determine whether the integral converges or diverges. If it converges, find the value.

a) $\int_1^{\infty} \frac{dx}{x^2}$

b) $\int_1^{\infty} \frac{dx}{x}$

c) $\int_1^{\infty} \frac{dx}{x-1}$

d) $\int_0^1 \frac{dx}{x^2}$

e) $\int_0^1 \frac{dx}{\sqrt{x}}$

f) $\int_1^3 \frac{dx}{x}$

7. A metal sphere of radius $a$ has electric charge $q > 0$. Let $r$ be the distance from the center of the sphere to a point in space. It is known from electrostatic theory that the induced electric potential is $V(r) = \frac{q}{2\pi} \int_0^a \frac{dx}{(r^2-2rx+a^2)^{1/2}}$.

a) Evaluate $V(r)$. Consider two cases: $0 \leq r \leq a$ and $r > a$.

b) Sketch the graph of $V(r)$ for $r \geq 0$.

8. An aquarium full of water is 2 m long, 0.5 m wide, and 1 m high. Find the work done in pumping the water out the top of the aquarium. If the width of the aquarium is doubled, is the work also doubled? If the height is doubled, is the work also doubled?

9. Two identical ions repel each other with force $F = -\frac{q^2}{4\pi\epsilon_0 r^2}$, where $q$ is the ion charge, $r$ is the distance between the ions, and $\epsilon_0$ is the free-space permittivity. The negative sign indicates a repulsive force. (a) An ion is held fixed at $x = 0$. Find the work done in moving another ion from $x = 3$ to $x = 2$. (b) An ion is held fixed at $x = 1$. Find the work done in moving another ion from $x = 3$ to $x = 2$. (c) Two ions are held fixed at $x = 0$ and $x = 1$. Find the work done in moving a third ion from $x = 3$ to $x = 2$. (d) A metal rod of uniform charge density is held fixed on the interval $0 \leq x \leq 1$. The total charge on the rod is $q$. Find the work done in moving an ion from $x = 3$ to $x = 2$.

10. A cable hanging between two poles has the shape $y = \cosh x$, $-1 \leq x \leq 1$. a) Find the arclength of the cable. b) Find the surface area obtained by rotating the cable about the $x$-axis.

11. Sketch the region in the $xy$-plane and find the center of mass.

a) $\{(x, y) : 0 \leq y \leq x^2, \ 0 \leq x \leq 2\}$

b) $\{(x, y) : x^2 \leq y \leq 4, \ 0 \leq x \leq 2\}$

c) $\{(x, y) : -1 \leq x \leq 1, \ 0 \leq y \leq 1 - x^2\}$

d) $\{(x, y) : 0 \leq y \leq \frac{1}{1+x^2}, \ 0 \leq x < \infty\}$
12. The lifetime of a light bulb is described by an exponential distribution with mean 1000 hours. Find the probability that the light bulb: a) fails in the first 200 hours, b) lasts more than 800 hours.

13. Let \( f(x) = \frac{1}{\pi \sqrt{x(1-x)}} \) for \( 0 < x < 1 \) and zero otherwise. Sketch the graph and show that \( f(x) \) is a valid pdf.

differential equations

14. Find the solution of the differential equation with initial condition \( y(0) = y_0 \). Sketch the solution for \( t \geq 0 \). Find \( \lim_{t \to \infty} y(t) \).

a) \( y' = -2y \), \( y_0 = 1 \)  

b) \( y' = 1 - 2y \), \( y_0 = 0 \)  

c) \( y' = 1 - y^2 \), \( y_0 = 0 \)  

d) \( y' = -ty \), \( y_0 = 1 \)

15. Consider the differential equation \( y'' = y \).

a) Show that \( y(t) = c_1 e^t + c_2 e^{-t} \) is a solution, where \( c_1, c_2 \) are arbitrary constants.

b) Find the solution \( y(t) \) subject to the initial conditions \( y(0) = 1 \), \( y'(0) = 0 \).

c) Repeat part (b) for initial conditions \( y(0) = 0 \), \( y'(0) = 1 \).

16. The cell count in a bacteria culture grows at a rate proportional to its size. After 30 minutes there are 200 cells and after 90 minutes there are 800 cells. (The answers should be expressed as integers.)

a) Find the initial cell count.  
b) When will the cell count reach 6400?

17. Polonium-214 has a half-life of \( 1.4 \times 10^{-4} \) s. If a sample has initial mass 40 mg, how long will it take for the mass to decay to 30 mg?

18. A tiger consumes 2500 calories per day and expends 20 calories per kg of its mass per day in daily activity. Assume that 1 kg of the tiger’s mass is equivalent to 10,000 calories. Formulate a differential equation for the mass of the tiger as a function of time, where \( y(t) \) denotes the tiger’s mass (kg) as a function of time \( t \) (day). Solve the equation subject to a general initial condition \( y(0) = y_0 \). What value does the tiger’s mass approach as time increases? Sketch the graph of the tiger mass \( y(t) \) for \( t \geq 0 \). Consider three cases, \( y_0 = 100, 125, 150 \), and sketch all three solutions on the same plot.

19. A thermometer at room temperature 70°F is placed in a patient’s mouth. After one minute the thermometer reads 95°F and after two minutes it reads 100°F. Find the patient’s temperature.

20. In a common model for an epidemic, the rate of change of the infected population is proportional to the product of the number of people currently infected and the number of people not yet infected. In a town with 4000 inhabitants, if 10 people are infected at the beginning of the week and 20 people are infected at the end of the week, how long does it take for half the population to be infected?

21. Consider the differential equation \( y' = 2y \) with initial condition \( y_0 = 1 \). We are interested in the solution at time \( t = 1 \). Let \( u_n \) be the numerical solution given by Euler’s method after \( n \) time steps with \( \Delta t = \frac{1}{n} \). Find the expression for \( u_n \) and evaluate the limit \( \lim_{\Delta t \to 0} u_n \).

series

22. Determine whether the series converges or diverges. Justify your answer.

a) \( \sum_{n=1}^{\infty} \frac{1}{2n} \)  
b) \( \sum_{n=1}^{\infty} \frac{1}{2^n} \)  
c) \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)  
d) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \)  
e) \( \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2} \)
23. Express the repeating decimal as a rational number (i.e. a ratio of two integers).
   a) 0.1111111111...  b) 0.1212121212...  c) 0.4999999999...

24. Find the sum of the series.  a) \( \sum_{n=0}^{\infty} \frac{2^n}{n!} \)  b) \( \sum_{n=1}^{\infty} \frac{1}{3^n} \)  c) \( \sum_{n=1}^{\infty} \frac{n}{3^n} \)

25. It is known that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \). Use this to evaluate \( \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \).

26. For each series, find a bound for \( |s - s_{10}| \) using the estimates derived in class.
   a) \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)  b) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \)

27. Two students walk towards each other at 2 mi/hr starting from a separation of 20 miles. At the same time, a dog starts running back and forth between the students at 10 mi/hr. Let \( D \) be the total distance the dog has traveled when the students finally meet. Express \( D \) as an infinite series and find the sum of the series.

28. Winning a game of ping-pong requires a lead of two points, i.e. if the final score is tied, you must score two consecutive points in order to win the game. Suppose your probability of scoring a point is \( p \), where \( 0 < p < 1 \). If the final score is tied, find the probability you will eventually win the game. Evaluate for \( p = \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \). Interpret.

29. Start with the closed interval \([0, 1]\). Remove the open interval \(( \frac{1}{3}, \frac{2}{3} )\). That leaves the two intervals \([0, \frac{1}{3}]\) and \([\frac{2}{3}, 1]\). Remove the middle third of each of those. That leaves four intervals. Remove the middle third of each of those. Continue the process indefinitely. The Cantor set is the set of all points remaining after all the intervals have been removed.
   a) Show that the total length of all the intervals removed is 1.
   b) Show that, nonetheless, the Cantor set contains infinitely many numbers.

**power series, Taylor series**

30. Find the interval of convergence of the power series. Find the function \( f(x) \) that is represented by the power series. Sketch the graph of \( f(x) \) and indicate the interval of convergence on the \( x \)-axis.
   a) \( \sum_{n=0}^{\infty} x^n \)  b) \( \sum_{n=0}^{\infty} \frac{x^n}{2^n} \)  c) \( \sum_{n=0}^{\infty} (x - 1)^n \)  d) \( \sum_{n=1}^{\infty} \frac{x^n}{n} \)  e) \( \sum_{n=1}^{\infty} nx^n \)

31. Find the power series representation for \( f(x) = \frac{1}{1-x} \) about \( x = \frac{1}{2} \).

32. Find the Taylor series for \( \sin x \) and \( \cosh x \) about \( x = 0 \).

33. a) By squaring and adding the Taylor series for \( \sin x \) and \( \cos x \), find the Taylor series for \( \sin^2 x + \cos^2 x \), up to the \( O(x^6) \) term. b) Could you have predicted the answer to part (a) without squaring and adding the series?

34. Find the Taylor series for \( f(x) = e^{-x^2} \) about \( x = 0 \). Sketch \( f(x), T_0(x), T_1(x), T_2(x) \) in a neighborhood of \( x = 0 \). Label each curve.

35. Let \( f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases} \) Evaluate the following limits.
   a) \( \lim_{x \to 0^-} f(x) \)  b) \( \lim_{x \to 0^+} f(x) \)  c) \( \lim_{x \to 0^+} f'(x) \)  d) \( \lim_{x \to 0^+} f''(x) \)  e) Sketch the graph of \( f(x) \).

36. Find an approximate value for \( \sqrt{10} \) which is accurate to within 0.005.

37. Use the Taylor series for \( f(x) = \ln(1 + x) \) about \( x = 0 \) to evaluate \( \ln \frac{3}{2} \) to within \( 10^{-3} \).

38. Find the first two nonzero terms in the Taylor series for \( f(x) \) about \( x = 0 \).
a) $e^{-x} \sin x$  
b) $(1 - \cos x)/x$  
c) $\tan x$  
d) $\tan^{-1} x$

39. The Bernoulli numbers $B_n$ are defined by $\frac{xe^x - 1}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$. Find $B_0, B_1, B_2$.

40. Show that the following functions satisfy $f(0) = 0, f'(0) = 1$. Find $f''(0)$ in each case. If the functions are graphed in a neighborhood of $x = 0$, in what order do they appear (from top to bottom)?  
a) $x$  
b) $\sin x$  
c) $\ln(1 + x)$  
d) $e^x - 1$

41. Recall the Bessel function of order zero, $J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}$.

a) Evaluate $\int_0^1 J_0(x) \, dx$ using 2 terms in the series. Find an upper bound for the error.

b) Show that $J_0(x)$ satisfies the differential equation $xy'' + y' + xy = 0$.

42. Let $f(t) = \sum_{n=0}^{\infty} t^n$.

a) Show that $f(t)$ satisfies the differential equation $y' = y^2$ with initial condition $y(0) = 1$.

b) Solve the differential equation for $f(t)$ by separation of variables. Do you recognize the result?

43. a) Show that $\int_0^{\infty} \frac{\sin x}{x} \, dx$ converges, but $\int_0^{\infty} \left| \frac{\sin x}{x} \right| \, dx$ diverges.

(hint: sketch the graph, express the integral as a series, bound the terms in the series)

b) Find the value of $\int_0^{\infty} \frac{\sin x}{x} \, dx$. (hint: let $f(a) = \int_0^{\infty} \frac{\sin x}{x} e^{-ax} \, dx$ for $a \geq 0$, evaluate $\lim_{a \to \infty} f(a)$, then $f'(a)$, then $f(a)$, and finally $f(0)$)

44. Use the 1st order Taylor approximation for $\cos x$ about $x = 0$ to show that $|\cos \frac{\pi}{5} - 1| \leq \frac{1}{2} (\frac{\pi}{5})^2$. Derive a more accurate result using the 3rd degree Taylor approximation.

45. Recall the error function, $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$. Find the first three terms in the Taylor series for $\text{erf}(x)$ about $x = 0$.

46. a) expand $\frac{a}{a+b}$ in powers of $\frac{a}{b}$  
b) expand $\sqrt{R^2 - r^2}$ in powers of $\frac{r}{R}$

Find the first three nonzero terms in each expansion.

47. Show that $f(x + h) = f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \frac{1}{3!} f'''(x)h^3 + \cdots$. This is an alternative form of the Taylor series.

48. The equation $\frac{x^2}{(1 + \epsilon)^2} + y^2 = 1$ defines an ellipse in the $xy$-plane (assume $0 \leq \epsilon < 1$).

a) Find the intercepts on the $x$-axis and $y$-axis. Sketch the ellipse.

b) Let $A(\epsilon)$ be the area of the ellipse. Express $A(\epsilon)$ as a definite integral.

c) Find the first 2 nonzero terms in the power series expansion of $A(\epsilon)$ about $\epsilon = 0$.

49. The gravitational potential energy function due to a pair of point masses $m_1, m_2$ located at $x_1, x_2$ on the $x$-axis is $V(x) = \frac{Gm_1}{|x - x_1|} + \frac{Gm_2}{|x - x_2|}$, where $G$ is the gravitational constant. For $x \to \infty$, the potential energy function can be approximated by $V(x) \approx \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \cdots$, where $a, b, c, \ldots$ are constants that depend on $m_1, m_2$ and $x_1, x_2$. Find the values of $a, b, c$. Are any of the results familiar?
50. The Lennard-Jones potential energy function, \( V(r) = V_0 \left( \frac{r_0}{r} \right)^{12} - 2 \left( \frac{r_0}{r} \right)^{6} \), describes the interaction between two particles (e.g. atoms, molecules), where \( V_0 \) and \( r_0 \) are positive constants, and \( r \geq 0 \) is the distance between the particles.

a) Find \( \lim_{r \to 0} V(r) \), \( \lim_{r \to \infty} V(r) \).

b) Show that \( V(r) \) has a minimum at \( r = r_0 \).

c) Sketch the graph of \( V(r) \) for \( r \geq 0 \).

d) Find \( T_2(r) \), the quadratic Taylor approximation for \( V(r) \) at \( r = r_0 \).

e) Find the work done in separating two particles from \( r = r_0 \) to \( r = \infty \) (this corresponds to dissociating a molecule). The force is \( f(r) = -V'(r) \).

51. Use the 2nd degree Taylor approximation of \( \sqrt{1 + x^2} \) at \( x = 0 \) to approximate \( \int_0^1 \sqrt{1 + x^2} \, dx \).

Find an upper bound for the error.

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**binomial series**

52. Recall the binomial expansion, \((a + b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n}b^n\), where \( k \geq 1 \) is an integer.

a) Show that \( \binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1} \).

b) Explain the connection between the formula in (a) and Pascal’s triangle shown below.

\[
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
\end{array}
\]

c) Fill in the next two rows of the triangle. Use this to expand \((a + b)^6\).

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**complex numbers, polar coordinates**

53. Express the complex number in Cartesian form \( x + iy \) and polar form \( re^{i\theta} \). Plot each number in the complex plane.  
a) \( 1 + i \)  
b) \((1 + i)^2\)  
c) \((1 + i)^3\)  
d) \( \frac{1}{1+i} \)  
e) \( \sqrt{1+i} \)

54. Compute \((1 + i)^6\) two ways, using: (a) binomial expansion , (b) polar form.

55. Find the roots of the equation. Plot the roots in the complex plane.

a) \( z^2 + 2z - 2 = 0 \)  
b) \( z^2 + 2z + 2 = 0 \)  
c) \( z^2 = 1 \)  
d) \( z^3 = 1 \)  
e) \( z^4 = 1 \)  
f) \( z^5 = 1 \)

56. a) Show that \( (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \).

b) Use this to derive the double-angle formulas, \( \sin 2\theta = 2 \sin \theta \cos \theta \), \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta \).

57. Derive the addition formulas for sine and cosine, shown below. (hint : \( e^{i(a+b)} = e^{ia} \cdot e^{ib} \))

\[
\cos(a + b) = \cos a \cos b - \sin a \sin b , \quad \sin(a + b) = \sin a \cos b + \cos a \sin b
\]

58. a) Use integration by parts to find the antiderivative. \( \int e^{ax} \cos bx \, dx \), \( \int e^{ax} \sin bx \, dx \)

b) Show that \( e^{(a+ib)x} = e^{ax} \cos bx + ie^{ax} \sin bx \) and \( \int e^{(a+ib)x} \, dx = \frac{a-ib}{a^2+b^2} e^{(a+ib)x} \).

c) Take the real and imaginary parts in (b) to rederive the formulas obtained in (a).

59. Derive the following results using Euler’s formula, \( e^{ix} = \cos x + i \sin x \).

a) \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \)  
b) \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \)

Derive the following formulas using (a) and (b).

c) \( \frac{d}{dx} \cos = -\sin x \)  
d) \( \frac{d}{dx} \sin = \cos x \)  
e) \( \sin^2 x + \cos^2 x = 1 \)  
f) \( \sin 2x = 2 \sin x \cos x \)

g) \( \cos 2x = \cos^2 x - \sin^2 x \)  
h) \( \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta \)