6. Prove. a) \[ \lim_{x \to a} f(x) = L \] then \[ \lim_{x \to a} f(x) = L \]

5. Find the antiderivative. a) \[ \int x^n \, dx = \frac{x^{n+1}}{n+1} \] (n \neq -1)

4. Evaluate the limit by any means. a) \[ \lim_{x \to 0} x^2 \]

3. Derive the formula using the FTC. a) \[ \int f(x) \, dx = F(x) + C \]

2. Express the integral as a limit of Riemann sums and evaluate the limit. Check your answer section 1.2 area, 1.3 definite integral, 1.4 FTC

1. True or False. Justify your answer with a reason or counterexample.

a) \[ \sum_{i=1}^{12} 2i = 156 \]

b) \[ \sum_{i=1}^{12} \left( \frac{1}{i} - \frac{1}{i+1} \right) = \frac{12}{13} \]

c) \[ \sum_{n} (n-i)^2 = \sum_{n} i^2 \]

f) If \[ \int_{a}^{b} f(x) \, dx > 0 \], then \[ f(x) > 0 \] for \[ a \leq x \leq b \].

g) If an integral \[ \int_{a}^{b} f(x) \, dx \] is computed using the right-hand Riemann sum and the number of intervals \( n \) is doubled, then the error is approximately also doubled.

h) \[ \frac{d}{dx} \int_{0}^{x} \sqrt{1+t^2} \, dt = \sqrt{1+x^2} \]

i) \[ \int_{0}^{\infty} e^{-x} \cos x \, dx = \int_{0}^{\infty} e^{-x} \sin x \, dx \]

j) A spring has natural length 10 cm. If 2 J of work are needed to stretch the spring from length 10 cm to 15 cm, then 4 J of work are needed to stretch it from length 10 cm to 20 cm.

k) \[ \int_{1}^{\infty} \frac{dx}{x} \] is a proper integral because \[ \frac{1}{x} \] is a bounded function for \( 1 \leq x < \infty \).

l) If \[ \lim_{x \to \infty} f(x) = 0 \], then the improper integral \[ \int_{1}^{\infty} f(x) \, dx \] converges.

m) The area under the graph of \( y = \frac{1}{x^2} \) from \( x = 1 \) to \( x = \infty \) is finite.

n) If \( 0 \leq f(x) \leq g(x) \) for \( x \geq 1 \) and \( \int_{1}^{\infty} g(x) \, dx \) converges, then \( \int_{1}^{\infty} f(x) \, dx \) also converges.

o) The error function, defined by \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \, dt \), satisfies \( \text{erf}''(0) = \text{erf}(0) \).

p) \[ \int_{0}^{\pi/2} \sin^2 x \, dx = \int_{0}^{\pi/2} \cos^2 x \, dx \]

section 1.2 area, 1.3 definite integral, 1.4 FTC

2. Express the integral as a limit of Riemann sums and evaluate the limit. Check your answer using the FTC. a) \[ \int_{0}^{2} x \, dx \]

b) \[ \int_{0}^{1} x^3 \, dx \]

c) \[ \int_{0}^{1} e^{-x} \, dx \]

3. Derive the formula \[ \int_{a}^{b} x^2 \, dx = \frac{1}{3}(b^3 - a^3) \] by Riemann sums.

4. Evaluate the limit by any means.

a) \[ \lim_{x \to \infty} xe^{-x} \]

b) \[ \lim_{n \to \infty} \sum_{i=1}^{n} \left( 1 + \frac{i}{n} \right)^3 \frac{1}{n} \]

c) \[ \lim_{x \to 1} \int_{0}^{x} f(t) \, dt \]

d) \[ \lim_{x \to 0} \frac{e^x - 1}{x} \]

e) \[ \lim_{r \to 1} \frac{1}{1-r^{11}} \]

5. Find the antiderivative.

a) \[ \int xe^{-x^2} \, dx \]

b) \[ \int x^2 e^{-x} \, dx \]

c) \[ \int x \sin x \, dx \]

d) \[ \int \frac{dx}{\sqrt{4-x^2}} \]

e) \[ \int \sqrt{4-x^2} \, dx \]

6. Prove. a) \[ \frac{1}{20} \leq \int_{0}^{1} \frac{x^9}{1+x} \, dx \leq \frac{1}{10} \]

b) \[ \int_{0}^{1} x (1-x)^{11} \, dx = \frac{1}{156} \]
7. A cable of length $L$ m hangs from the top of a building. The cable has cross-sectional area $A$ m$^2$ and density $\rho$ kg/m$^3$. a) Find the work done in pulling the cable to the top of the building. b) If the length of the cable is doubled, is the work also doubled?

8. A force of 30 N is needed to stretch a spring from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

9. Two ions of charge $q$ repel each other with force $f(r) = -\frac{q^2}{4\pi\varepsilon_0 r^2}$ N, where $\varepsilon_0$ is the vacuum permittivity and $r$ is the distance between the ions measured in meters. If one ion is held fixed at $x = 0$ mm, find the work done in moving the second ion from $x = 3$ mm to $x = 2$ mm.

10. A pyramid is built of stone with density $\rho$ kg/m$^3$. The base of the pyramid is a square, and the vertex is directly above the center of the base. The length of a side of the base is $L$ m and the height to the vertex above the base is $H$ m. a) Derive a formula for the work done in building the pyramid (i.e. raising the stone from ground level to its level in the pyramid). b) If the length $L$ and height $H$ are doubled, by what factor does the work increase? c) Which requires more work, building the lower half or the upper half of the pyramid?

11. Determine whether the integral converges or diverges. If it converges, find the value. If it diverges, give a reason.
   a) $\int_1^\infty \frac{dx}{x^4}$  
   b) $\int_0^\infty x^2 e^{-x} dx$  
   c) $\int_0^\infty e^{-x} \sin x dx$  
   d) $\int_1^\infty \left( \frac{1}{x} - \frac{1}{x + 1} \right) dx$  
   e) $\int_{-r}^{r} \sqrt{r^2 - x^2} dx$  
   f) $\int_{-r}^{r} \frac{dx}{\sqrt{r^2 - x^2}}$  
   g) $\int_1^\infty \frac{dx}{1 + x^2}$  
   h) $\int_1^\infty \frac{dx}{\sqrt{1 + x^2}}$  
   i) $\int_1^\infty \frac{x}{\sqrt{1 + x^2}} dx$  
   j) $\int_1^\infty \frac{dx}{x^2 - 1}$

12. Show that $\int_0^\infty \frac{\ln x}{1 + x^2} dx = 0$.  
   hint: substitute $u = \frac{1}{x}$

13. A patient receives an intravenous drug at the rate $r(t) = 2te^{-2t}$ ml/sec, where $t$ is the time in seconds since the treatment started.
   a) Find the total dose the patient receives in the limit $t \to \infty$.
   b) What fraction of the total dose is received in the first 5 seconds?

14. Find the arclength of the curve on the interval $0 \leq x \leq 1$.
   a) $y = \sqrt{1 - x^2}$  
   b) $y = \int_0^x \sqrt{1 - t^2} dt$  
   c) $y = \frac{e^x + e^{-x}}{2}$  
   d) $y = \sqrt{x^3}$  
   e) $y = 2x^2$

15. Sketch the curve $y = \sqrt{2x - x^2}$ for $0 \leq x \leq 2$ and find its arclength.
16. Sketch the curve $y = \sqrt{x}$ for $0 \leq x \leq 1$ and find its arclength. (hint: substitute $y = \sqrt{x}$ in the arclength integral)

miscellaneous

17. Sketch each function on the interval $0 \leq x \leq 2\pi$.
   a) $\cos x$  
   b) $\cos 2x$  
   c) $\frac{1}{2}\cos 2x$  
   d) $\frac{1}{2} + \frac{1}{2}\cos 2x$  
   e) $\cos^2 x$