Theorem 4.2: If $R$ is an integral domain and $f(x), g(x) \in R[x]$ are nonzero, then
$$\deg f(x)g(x) = \deg f(x) + \deg g(x).$$

Corollary 4.3: If $R$ is an integral domain, then $R[x]$ is an integral domain.
Corollary 4.4: If $R$ is a ring and $f(x), g(x), f(x)g(x) \in R[x]$ are nonzero, then
$$\deg f(x)g(x) \leq \deg f(x) + \deg g(x).$$

Corollary 4.5: Let $R$ be an integral domain and $f \in R[x]$. The polynomial $f(x)$ is a unit iff $f(x)$ is a constant polynomial that is a unit in $R$.

Theorem 4.6: (The Division Algorithm) Let $F$ be a field and $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Then there exist unique polynomials $q(x), r(x)$ such that
$$f(x) = g(x)q(x) + r(x)$$
and either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Theorem 4.7: Let $F$ be a field with $a(x), b(x) \in F[x]$ with $b(x) \neq 0$. If $b(x) | a(x)$, then $cb(x) | a(x)$ for each nonzero $c \in R$.

Theorem 4.8: Let $F$ be a field with $a(x), b(x) \in F[x]$ not both zero. Then there is a unique gcd $d(x)$ of $a(x)$ and $b(x)$. Moreover, there exist polynomials $u(x), v(x)$ such that
$$d(x) = a(x)u(x) + b(x)v(x).$$

Theorem 4.10: Let $F$ be a field and $a(x), b(x), c(x) \in F[x]$. If $a(x) | b(x)c(x)$ and $a(x)$ and $b(x)$ are relatively prime, then $a(x) | c(x)$.

Theorem 4.11: A nonzero polynomial is reducible in $F[x]$ iff it can be written as the product of two polynomials of lower degree.

Theorem 4.12: Let $p(x) \in F[x]$ be nonconstant. TFAE:
1. $p(x)$ is irreducible.
2. If $b(x)$ and $c(x)$ are any polynomials such that $p(x)|b(x)c(x)$, then $p(x)|b(x)$ or $p(x)|c(x)$.
3. If $r(x)$ and $s(x)$ are any polynomials such that $p(x) = r(x)s(x)$, then $r(x)$ or $s(x)$ is a nonzero constant polynomial.

Theorem 4.14: Polynomials in $F[x]$ factor uniquely up to reordering and multiplication by units.

Theorem 4.15: (The Remainder Theorem) The remainder when a polynomial $f(x) \in F[x]$ is divided by $(x - a)$ equals $f(a)$.

Theorem 4.16: (The Factor Theorem) An element $a \in F$ is a root of $f(x) \in F[x]$ iff $(x - a)$ is a factor of $f(x)$ in $F[x]$.

Theorem 4.21: (Rational Root Test) If the rational number $r/s \neq 0$ is a root of $f(x) = a_nx^n + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$, then $r|a_0$ and $s|a_n$. 

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Theorem 4.24: (Eisenstein’s Criterion) If \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x] \) and a prime \( p \) divides each of \( a_0, a_1, \ldots, a_n - 1 \), but \( p \) does not divide \( a_n \) and \( p^2 \) does not divide \( a_0 \), then \( f(x) \) is irreducible in \( \mathbb{Q}[x] \).

Theorem 4.26: (The Fundamental Theorem of Algebra) Every non-constant polynomial in \( \mathbb{C}[x] \) has a root in \( \mathbb{C} \).

Theorem 5.1: The relation of congruence modulo a nonzero \( p(x) \in F[x] \) is an equivalence relation.

Theorem 5.2: For a nonzero \( p(x) \in F[x] \), if \( f(x) \equiv g(x) \mod p(x) \) and \( h(x) \equiv k(x) \mod p(x) \), then \( f(x) + h(x) \equiv g(x) + k(x) \mod p(x) \) and \( f(x)h(x) \equiv g(x)k(x) \mod p(x) \).

Theorem 5.3: The congruence \( f(x) \equiv g(x) \mod p(x) \) holds iff \( [f(x)] = [g(x)] \in F[x]/p(x) \).

Theorem 5.6: For nonconstant \( p(x) \in F[x] \), if \( [f(x)] = [g(x)] \) and \( [h(x)] = [k(x)] \) in \( F[x]/p(x) \), then \( [f(x)+h(x)] = [g(x)+k(x)] \) and \( [f(x)h(x)] = [g(x)k(x)] \).

Theorem 5.7: For nonconstant \( p(x) \in F[x] \), the set \( F[x]/p(x) \) is a commutative ring with 1. Moreover, the ring \( F[x]/p(x) \) contains a subring isomorphic to \( F \).

Theorem 5.9: For nonconstant \( p(x) \in F[x] \), if \( f(x) \in F[x] \) is relatively prime to \( p(x) \), then \( [f(x)] \) is a unit in \( F[x]/p(x) \).

Theorem 5.10: For nonconstant \( p(x) \in F[x] \), TFAE:

1. \( p(x) \) is irreducible in \( F[x] \)
2. \( F[x]/p(x) \) is a field
3. \( F[x]/p(x) \) is an integral domain.

Theorem 5.11: If \( p(x) \) is irreducible in \( F[x] \), then \( F[x]/p(x) \) is a field extension of \( F \) that contains a root of \( p(x) \).