Midterm Exam October 17, 2002

Instructions. 1. Two sides of a 3.5in. x 5in. sheet of notes from home. Closed book.
2. Show work and explain clearly.
3. There are six questions, one per page. 100 points total

1. (15 points) Show that if $f(z)$ is an analytic function on a connected open set $\Omega$ and $f$ depends only on $x$ where $z = x + iy$, then $f$ is a constant.
2. (14 points) Suppose that \( f(z) = 1/z \), \( g(z) = z^2 \) and \( A = \{z : 1 < |z| < 2, \ 0 < \arg z < \pi/4 \} \). The set \( A \) is sketched below. Sketch the regions \( f(A) = \{f(z) : z \in A \} \) and \( g(A) = \{g(z) : z \in A \} \) on the axes provided. Indicate important lengths and angles in the sketch.
3. (8+6 points) a. At what point(s) of the closed disc with center at \( i \) and radius equal to 2 does the modulus of the analytic function \( f(z) = z^4 \) attain its maximum and minimum values.

b. Explain why this is consistent with the maximum and minimum modulus principles for analytic functions.
4. (10+10+10 points) Using theorems of complex analysis, evaluate the following three integrals. Be sure to state the general formula you are applying and to what function(s).

a. \[
\frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{(z-2)(z-4)^3} \, dz
\]

b. \[
\frac{1}{2\pi i} \oint_{\partial \Omega} \frac{1}{(z-2)(z-4)^3} \, dz,
\]
   where \( \Omega \) is the annular region \( 1 \leq |z| \leq 3 \). As usual, \( \partial \Omega \) denotes the boundary of \( \Omega \) oriented in the standard sense.

c. \[
\frac{1}{2\pi i} \oint_{\partial R} \frac{1}{(z-2)(z-4)^3} \, dz,
\]
   where \( R \) is the annular region \( 3 \leq |z| \leq 5 \).
5. (5+5+5 points) Liouville’s Theorem asserts that an analytic function which is defined on the entire complex plane and is bounded must be a constant function. Does this result imply that any of the following functions is constant? Explain.

a. \( \frac{1}{1 + x^2 + y^2} \),

b. \( \frac{1}{z^2} \),

c. \( \sin z \).
6. (6+6 points) a. The equation \( z(t) = 6 e^{it}, \ 0 \leq t \leq 1 \) is a parametric equation a curve. Sketch it. Be sure to label important points, coordinates, angles, distances, ... etc.

b. Sketch a domain in the complex \( z \) plane which is mapped in a one to one way onto the upper half plane \( \{ w : \text{Im} \ w > 0 \} \) by the mapping \( w = e^z \).