Group Velocity at Smooth Points of Hyperbolic Characteristic Varieties

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Dedication. To my friend Jean-Michel Bony with best wishes and appreciation for what he has taught me of mathematics other things.

Suppose that $P(D)$ is a homogeneous hyperbolic polynomial of degree $m \geq 1$ with time-like covector $\theta$. Here $D = \partial/\partial y$ with $y \in \mathbb{R}^n$. The symbol $P(\eta)$ is a homogeneous polynomial on $(\mathbb{R}^n)^*$. Hyperbolicity with respect to $\theta \in (\mathbb{R}^n)^*$ means that for any $\eta \in (\mathbb{R}^n)^*$ the equation

$$P(\eta + s\theta) = 0$$

has only real roots $s$. In particular, $P(\theta) \neq 0$.

The characteristic variety

$$\text{Char } P := \{ \eta \in \mathbb{R}^n \setminus 0 : P(\eta) = 0 \}$$

is a conic real algebraic variety in $(\mathbb{R}^n)^*$. Since the equation (1) has $m$ complex roots (counting multiplicity), and they all are real, it follows that every real line $\eta + s\theta$ intersects the variety in at least one point and no more than $m$ points which shows that the variety has codimension 1 in $(\mathbb{R}^n)^*$. The fundamental stratification of real algebraic geometry (see [BR]) asserts that except for a set of codimension at least 2, the variety consists of smooth points, that is points where locally the variety is equal to the zero set of a real analytic function with nonvanishing gradient.

Definitions. If $\eta \neq 0$ is a point of the characteristic variety then $Q_\eta(\eta)$ is the homogeneous polynomial of degree $k \geq 1$ which is the leading term in the expansion of $P(\eta + \eta)$ about $\eta$.

$$P(\eta + \eta) = Q_\eta(\eta) + \text{higher order terms in } \eta, \quad Q_\eta \neq 0.$$ 

At a smooth point $\eta$, the annihilator of the tangent space $T_\eta(\text{Char } P)$ is a one dimensional linear subspace $L_\eta \in (T_\eta(\text{Char } P))^* = \mathbb{R}^n$. The lines in $\mathbb{R}^n$ parallel to $L_\eta$ are those moving with the group velocity (see [AR]).

This velocity describes the propagation of wave packets, pulses, and singularities associated with the frequencies $(\mathbb{R} \setminus 0) \eta$.

For variable coefficient operators, the above computations are performed in the tangent and cotangent spaces at a fixed point and $P$ is the principal symbol at that point. They are pertinent for example for symmetric hyperbolic systems and points of the characteristic variety which are microlocally of constant multiplicity.

If $\eta \in \text{Char } P$ is a smooth point of multiplicity one, that is $P(\eta) = 0$ and $dP(\eta) \neq 0$, then $dP(\eta)$ is a basis for $L_\eta$ and one has a simple way of recovering the velocity from the symbol.

In an analogous way, at a smooth point one can write the variety locally as $q = 0$ with $dq \neq 0$, then $dq(\eta)$ is a basis for $L_\eta$. However, in real algebraic geometry it is not in general easy to find a function $q$ starting from the defining function $P$ when the roots have multiplicity greater than one. The following two results provide a straightforward algorithm to compute the group velocity for our hyperbolic operators.

* Partially supported by the US National Science Foundation grant NSF-DMS-0104096
**Theorem.** If \( \eta \) is a smooth point of the characteristic variety and \( Q_{\eta} \) is as above, then there is a linear form \( \ell(\eta) \) so that the tangent plane at \( \eta \) to the characteristic variety of \( P \) is equal to \( \{ \ell(\eta - \eta) = 0 \} \), and, \( Q_{\eta}(\eta) = \ell(\eta)^k \).

**Corollary.** If \( \eta \) is a smooth point of the characteristic variety and \( Q_{\eta} \) and \( L_{\eta} \) are as above, then for all \( \eta \) which are not in the characteristic variety of \( Q_{\eta} \) (e.g. \( \eta = \theta \)), \( dQ_{\eta}(\eta) \) is a basis for \( L_{\eta} \).

These results both rely on the fundamental theorems concerning Local Hyperbolicity (see [G]). That theory is closely related to the ideas of microhyperbolicity introduced by Bony and Shapira in [BS] (see [H, §8.7]).

The proof of the Theorem begins with the fact from [G] that \( Q_{\eta}(\eta) \) is hyperbolic with time-like covector \( \theta \). Then for every real \( \eta \) the equation

\[
Q_{\eta}(\eta + s\theta) = 0
\]

has only real roots \( s \).

**Lemma 1.** For every real \( \eta \) the equation (2) has exactly one root \( s \).

**Proof.** Since \( k \) is the degree of \( Q_{\eta} \), one has as \( \epsilon \to 0, \)

\[
e^{-kP(\eta + \epsilon(\eta + s\theta))} = Q_{\eta}(\eta + s\theta) + O(\epsilon).
\]

If (2) had two roots \( s_1 \) and \( s_2 \), then Rouche's theorem would imply that the characteristic variety of \( P \) would have points near \( \eta + \epsilon(\eta + s_j \theta) \) as \( \epsilon \to 0 \) violating the smooth variety hypothesis.

The next Lemma is then applied to \( R = Q_{\eta} \).

**Lemma 2.** If \( R(\eta) \) is a homogeneous polynomial hyperbolic with respect to the time-like covector \( \theta \) and for all real \( \eta \) the equation \( R(\eta + s\theta) = 0 \) has exactly one real root \( s \), then there is a linear form \( \ell(\eta) \) such that

\[
R(\eta) = \ell(\eta)^{\deg R}.
\]

**Proof.** Introduce coordinates \((\tau, \xi_1, \ldots, \xi_n-1)\) in \((\mathbb{R}^n)^*\) so that \( \theta = (1, 0, \ldots, 0) \). Then

\[
R(\tau, \xi) = R(1, 0, \ldots, 0) \left( \tau^k + a_1(\xi)\tau^{k-1} + \cdots + a_{k-1}(\xi)\tau + a_k(\xi) \right)
\]

with \( a_j(\xi) \) homogeneous of degree \( j \) and \( k = \deg R \geq 1 \).

By hypothesis, for each real \( \xi \) the equation \( R(\tau, \xi) = 0 \) has a unique root \( \tau = r(\xi) \). Therefore

\[
R(\tau, \xi) = R(1, 0, \ldots, 0)(\tau - r(\xi))^k.
\]

Equating coefficients of \( \tau^{k-1} \) shows that

\[
-k r(\xi) = a_1(\xi),
\]

so \( r(\xi) \) is a homogeneous polynomial of degree 1. The Lemma follows with \( \ell(\tau, \xi) = c(\tau - r(\xi)) \) provided that \( c \) is chosen to satisfy \( c^k = R(1, 0, \ldots, 0) \).
The constant $c$ and the functional $\ell$ are uniquely determined up to a factor which is a $k^{th}$ root of unity.

**Proof of Theorem.** Combining the above lemmas implies that $Q_{\eta}(\eta) = \ell(\eta)^k$. It remains to show that the tangent plane to the characteristic variety of $P$ is given by the equation $\ell(\eta - \eta) = 0$.

Introduce local coordinates $(\tau, \xi)$ as in the proof of Lemma 2. Since $\theta = (1, 0, \ldots, 0)$ is noncharacteristic for $P$, the variety of $P$ is given by the roots $\tau$ of $P(\tau, \xi) = 0$ with $\xi$ ranging over $\mathbb{R}^n \setminus 0$.

The points near $\eta = (\tau, \xi)$ are then given by the roots $\tau$ of

$$P(\tau + \epsilon \tau, \xi + \epsilon \xi) = 0,$$

with $|\xi| \leq 1$. Equation (2) takes the form

$$\epsilon^{-k} P(\tau + \epsilon \tau, \xi + \epsilon \xi) = Q_{\eta}(\tau, \xi) + O(\epsilon).$$

Since $Q_{\eta} = \ell^k$, the equation $Q_{\eta}(\tau, \xi) = 0$ is equivalent to the equation $\ell(\tau, \xi) = 0$. Since $\ell(\theta)^k = Q_{\eta}(\theta) \neq 0$, it follows that the solutions of $\ell(\tau, \xi) = 0$ are given by $\tau = \epsilon \xi$ for an appropriate $\tau$.

Rouché's Theorem applied to (5) shows that for $|\xi| < 1$ the roots of (4) are given by

$$\tau = \epsilon \xi + O(\epsilon).$$

The corresponding points $\eta = (\tau + \epsilon \tau, \xi + \epsilon \xi)$ of the characteristic variety of $P$ differ from $\eta$ by $O(\epsilon)$ and satisfy

$$\ell(\eta - \eta) = O(\epsilon^2).$$

This shows that the equation of the tangent plane is $\ell(\eta - \eta) = 0$. \hfill $\blacksquare$

**Proof of Corollary.** Since $Q_{\eta} = \ell^k$ one has

$$dQ_{\eta}(\eta) = k \ell(\eta)^{k-1} d\ell(\eta).$$

Since $\ell$ is a linear form on $(\mathbb{R}^n)^*$, $d\ell(\eta)$ is a vector which does not depend the point $\eta$ where the derivative is evaluated. The Theorem implies that $d\ell$ is a basis for $L_{\eta}$. Therefore, $dQ_{\eta}(\eta)$ is a basis whenever it is nonvanishing. This holds exactly for $\eta$ which satisfy $\ell(\eta) \neq 0$ which is exactly those $\eta$ which are not in the characteristic variety of $Q_{\eta}$. \hfill $\blacksquare$

**References**


