1. (Harnack’s inequality) Let \( u \in C^2 \) for \( |x| < a; u \in C^0 \) for \( |x| \leq a; u \geq 0, \Delta u = 0 \) for \( |x| < a \). Show that for \( |x| < a \),
\[
\frac{a^{n-2}(a-|x|)}{(a+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{a^{n-2}(a+|x|)}{(a-|x|)^{n-1}} u(0).
\]

2. Let \( G = G(y, x) \) be the Green’s function (for the Dirichlet problem for Laplace equation for domain \( \Omega \)). Show that
\[
K(y, x) = \frac{\partial}{\partial n_y} G(y, x) \geq 0, \quad \text{for } y \in \partial \Omega, x \in \Omega
\]

3. (a) Let \( \mu > 0 \) be a constant, \( u = u(x, t) \) be a positive solution of class \( C^2 \) of
\[
\begin{align*}
    u_t &= \mu u_{xx}, & \text{for } x \in \mathbb{R}, t > 0
\end{align*}
\]
Show that \( \theta = -2\mu u_x/u \) satisfies the viscous Burgers’ equation
\[
\theta_t + \theta \theta_x = \mu \theta_{xx} \quad \text{for } x \in \mathbb{R}, t > 0
\]
(b) For \( \phi \in C^2_0(\mathbb{R}) \) find a solution of (0.1) with initial value \( \theta(x, 0) = \phi(x) \), for which
\[
\lim_{t \to \infty} \theta(x, t) = 0
\]
(The viscose term \( \mu \theta_{xx} \) prevents singularities that would occur, compare with burgers’ equation \( \theta_t + \theta \theta_x = 0 \).)

4. Write down an explicit formula for a solution of
\[
\begin{align*}
    u_t - \Delta u + cu &= f & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u &= g & \text{on } \mathbb{R}^n \times \{t = 0\}
\end{align*}
\]

5. Given \( g : [0, \infty) \to \mathbb{R} \), with \( g(0) = 0 \), derive the formula for a solution of the initial boundary value problem
\[
\begin{align*}
    u_t - u_{xx} &= 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\
    u &= g & \text{on } \mathbb{R}_+ \times \{t = 0\} \\
    u &= 0 & \text{on } \{x = 0\} \times [0, \infty)
\end{align*}
\]
(Hint: Let \( v(x, t) = u(x, t) - g(t) \) and extend \( v \) to \( \{x < 0\} \) by odd reflection.
Solution \( u(x, t) = \sum_{n=0}^{\infty} \int_0^t \frac{1}{(n\pi)^2} e^{-\frac{n^2 \pi^2 t}{4}} g(s) \) ds.)