Let $U \subset \mathbb{R}^n$ be open, $k \in \mathbb{N} \cup \{0\}$.

1. Let $0 < \gamma \leq 1$. Show that $C^{k,\gamma}(U)$ is a Banach space.

2. Assume that $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\gamma \beta} \|u\|_{C^{0,1}(U)}^{1-\gamma \beta}.$$

In 3. and 4. $|D_k u| = (\sum_{|\alpha| = k} |\partial_\alpha x u|)^{1/2}$.

3. Integration by parts to prove the interpolation inequality

$$\int_U |Du|^2 \, dx \leq C(\int_U u^2 \, dx)^{1/2}(\int_U |D^2 u|^2 \, dx)^{1/2}$$

for all $u \in C^\infty(U)$. By approximation, prove this inequality for $u \in H^2(U) \cap H^1_0(U)$, where $H^2(U) := W^{2,2}(U)$, $H^1_0(U) := W^{1,2}_0(U)$.

4. Integration by parts to prove the interpolation inequality

$$\int_U |Du|^p \, dx \leq C(\int_U |u|^p \, dx)^{1/2}(\int_U |D^2 u|^p \, dx)^{1/2}$$

for $2 \leq p < \infty$ and all $u \in W^{2,p}(U) \cap W^{1,p}_0(U)$. (Hint: $\int_U |Du|^p \, dx = \sum_{i=1}^n \int_U u_{x_i} u_{x_i} |Du|^{p-2} \, dx$.)

5. Prove that $C^\infty_0(\mathbb{R}^n)$ is dense in $W^{k,p}(U)$, where $1 \leq p < \infty$, $U \subset \mathbb{R}^n$ is open, with $\partial U \subset C^1$. (Hint: we have already shown in class for $U$ bounded, you only need to show for unbounded open set $U \subset \mathbb{R}^n$.)

6. Let $U = \{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 
 1 - x_1 & \text{if } x_1 > 0, |x_2| < x_1 \\
 1 + x_1 & \text{if } x_1 < 0, |x_2| < -x_1 \\
 1 - x_2 & \text{if } x_2 > 0, |x_1| < x_2 \\
 1 + x_2 & \text{if } x_2 < 0, |x_1| < -x_2.
\end{cases} \quad (0.1)$$

For which $p$ does $u$ belong to $W^{1,p}(U)$?

7. Suppose $U$ is connected and $u \in W^{1,p}(U)$ satisfies

$$\nabla u = 0, \quad \text{a.e. in } U$$

Prove that $u$ is a constant a.e. in $U$. 