**Theorem 0.1 (L1-stability).** Let $u, v$ be piecewise $C^1$ entropy solutions of $u_t + f(u)_x = 0$. Then $||u(·, t) - v(·, t)||_{L^1}$ is a nondecreasing function of $t$.

**Proof.** (Kruzkov)

\[
\int |u(x, t) - v(x, t)| dx = \sum_n (-1)^n \int_{y_n(t)}^{y_n+1(t)} (u(x, t) - v(x, t)) dx,
\]

here, on $(y_n(t), y_{n+1}(t))$, $u - v$ has a definite sign, and on $(y_n, y_{n+1})$, $(y_{n+1}, y_{n+2})$, the sign of $u - v$ alternates.

Wants: \( \frac{d}{dt} (-1)^n \int_{y_n}^{y_{n+1}} (u - v) dx \leq 0 \).

**Lemma 1.** Let $a = a(t), b = b(t) \in C^1$. Assume $u$ is a piecewise $C^1$ weak solution of $u_t + f(u)_x = 0$. Then

\[
\frac{d}{dt} \int_{a(t)}^{b(t)} u(x, t) dx + f(u(b(t)) - t) - f(u(a(t)) + t) = b'(t)u(b(t)) - t - a'(t)u(a(t)) + t.
\]

**Proof.** Recall that weak solution means if $a < b$ are constants,

\[
\frac{d}{dt} \int_a^b u(x, t) dx + f(u(b, t)) - f(u(a, t)) = 0.
\]

Assume that $y = y(t) \in C^1$ such that $u$ is continuous on $\{x < y(t)\}, \{x > y(t)\}$. Then

\[
\frac{d}{dt} \int_{a(t)}^{b(t)} u(x, t) dx = \frac{d}{dt} \left( \int_{a(t)}^{y(t)} u(x, t) dx + \int_{y(t)}^{b(t)} u(x, t) dx \right) = b'(t)u(b(t)) - t - a'(t)u(a(t)) + t + \int_{a(t)}^{y(t)} u_l dx + \int_{y(t)}^{b(t)} u_r dx
\]

\[
= b'(t)u(b(t)) - t - a'(t)u(a(t)) + t + y'(t)u_l - y'(t)u_r + \int_{a(t)}^{y(t)} u_l dx + \int_{y(t)}^{b(t)} u_r dx
\]

\[
= b'(t)u(b(t)) - t - a'(t)u(a(t)) + t + y'(t)(u_l - u_r) - \int_{a(t)}^{y(t)} f(u)_l dx - \int_{y(t)}^{b(t)} f(u)_r dx
\]

\[
= b'(t)u(b(t)) - t - a'(t)u(a(t)) + t + y'(t)(u_l - u_r) - \int_{a(t)}^{y(t)} f(u)_l dx - \int_{y(t)}^{b(t)} f(u)_r dx
\]

\[
= b'(t)u(b(t)) - t - a'(t)u(a(t)) + t + f(u(a(t)) + t) - f(u(b(t)) - t)
\]

\[
= b'(t)u(b(t)) - t - a'(t)u(a(t)) + t + f(u(a(t)) + t) - f(u(b(t)) - t),
\]

here we use the Rankine-Hugoniot jump condition for the last equality. \( \square \)
Come back to the proof of the theorem. Assume without loss of generality that \((-1)^n = 1\). We consider three cases.

Case 1. \(u, v\) are continuous at \(y_n(t)\). Then \(y_n(t)\) is a characteristic for both \(u\) and \(v\) \((y_n(t) = a(u)t + s = a(v)t + s)\).

\[
\frac{d}{dt} \int_{y_n}^{y_{n+1}} (u(x,t) - v(x,t))dx = y_n'(t)(u(y_{n+1}(t)-t) - v(y_{n+1}(t)-t) - y_n'(t)(u(y_n(t),t) - v(y_n(t),t)) - f(u(y_{n+1}(t)-t)) + f(v(y_{n+1}(t)-t)) + f(u(y_n(t),t)) - f(v(y_n(t),t))
\]

\[
= y_n'(t)(u(y_{n+1}(t)-t) - v(y_{n+1}(t)-t) - f(u(y_{n+1}(t)-t)) + f(v(y_{n+1}(t)-t)))
\]

Case 2. \(v\) is continuous at \(y_{n+1}(t)\), \(u\) is discontinuous at \(y_{n+1}(t)\). We claim that \(y = y_{n+1}(t)\) is a shock curve for \(u\).

\[
u_t > v > u_t \text{ in a small neighborhood of } t \text{ (or } u_t < v < u_r\).
\]

\[
\frac{d}{dt} \int_{y_n}^{y_{n+1}} (u - v)dx = y_{n+1}'(t)(u_t - v) - f(u_t) + f(v_t) < 0,
\]

because of Oleinik entropy condition.

Case 3. \(u, v\) are discontinuous at \(y_{n+1}(t)\). We claim that \(y_{n+1}(t)\) is a shock curve for \(u\) in a neighborhood of \(t\). Indeed,

\[
\frac{d}{dt} \int_{y_n}^{y_{n+1}} (u - v)dx = y_{n+1}'(t)(u_t - v_t) - f(u_t) + f(v_t) < 0,
\]

because of \(u_t > v_t > u_r\).

\[\square\]

**Corollary 0.2.** \(u_0 \in L^1 \cap L^\infty\), then \(\exists!\) entropy solution.

Large time asymptotics.

\[u_0 \in L^\infty \cap L^1.\] Assume \(f'' > k > 0, f(0) = 0, f \in C^\infty.\) We derive a solution formula of Oleinik-Lax.

Assume \(u = u(x,t)\) is a piecewise \(C^1\) entropy solution of \(u_t + f(u)_x = 0, f'' > 0.\) Assume \(u(x,0) = 0\) for \(x < -M\) for some constant \(M > 0.\) Then \(u(x,t) = 0\) for \(x < -M(t)\) for some function \(M(t)\).

**Definition 0.3.** \(U(x,t) := \int_{-\infty}^{x} u(y,t)dy.\)

Then we have

\[U_t + f(U_x) = 0.\]

As \(f\) is convex, we have \(f(u) \geq f(v) + a(v)(u - v), \forall u, v.\) Equality holds exactly when \(u = v.\) Let \(v\) be fixed, take \(u = u(x,t),\) then

\[-U_t = f(U_x) \geq f(v) - a(v)v + a(v)U_x,
\]

\[a(v) = \frac{x - y}{t}.\] So

\[U_t + a(v)U_x \leq a(v)v - f(v), \forall v.\]

Since \(U_t + a(v)U_x = \partial_t U(a(v)t + y, t),\) we have

\[\partial_r (U(a(v)r + y, r)) \leq a(v)v - f(v).\]
Integrate from 0 to \( t \), we get

\[
U(x, t) - U(y, 0) \leq t(a(v)v - f(v)), \quad \forall \, v.
\]

Let \( g(v) := a(v)v - f(v) \). Then

\[
g'(a(v))a'(v) = a'(v)v + a(v) - f'(v) = a'(v)v.
\]

So \( g'(a(v)) = v, \quad \forall \, v \). So \( g' = a^{-1} := b \). Recall that \( a(v) = \frac{x - y}{t} \), so we have

\[
U(x, t) \leq U(y, 0) + tg\left(\frac{x - y}{t}\right), \quad \forall \, y.
\]

If we take \( v = u = u(x, t) \), then equalities hold. Then we have

**Theorem 0.4** (Lax-Oleinik). Assume \( f(0) = 0, f'' \geq k > 0 \). Assume that \( u \) is a piecewise \( C^1 \) entropy solution of \( u_t + f(u)_x = 0 \). Then \( u = b\left(\frac{x - y}{t}\right) \), where \( y \) minimizes

\[
G(x, y, t) := u(y, 0) + tg\left(\frac{x - y}{t}\right).
\]

Here \( a = f', b = a^{-1}, g' = b \). Assume that \( a(0) = c \), then \( g(c) = 0 \).

**Theorem 0.5** (Lax-Oleinik). Assume that \( u_0 \in L^1 \cap L^\infty \). Define \( u = b\left(\frac{x - y}{t}\right) \), where \( y = y(x, t) \) is one of the points that minimizes \( G(x, y, t) = U_0(y) + t g\left(\frac{x - y}{t}\right) \), here \( U_0(y) = \int_{x_0}^{y} u_0(x) dx \), i.e.,

\[
G(x, y(x, t), t) = \min_y G(x, y, t).
\]

Then \( u = u(x, t) \) is a solution of \( u_t + f(u)_x = 0 \) in distribution sense and all the discontinuities of \( u \) are shocks.

**Remark 0.6.** For fixed \( x, t \), \( G(x, y, t) \) has a minimum: \( g' = b, \ g''(z) = b'(z) = \frac{1}{a'(a^{-1}(z))} > 0 \), so \( g \) is convex, and then \( \min_y G(x, y, t) \) exists.

**Lemma 2.** Let \( y = y(x, t) \) be one of the points that minimize \( G(x, y, t) \). Then \( y(x, t) \) is non-decreasing of \( x \).

**Proof.** If \( x_1 < x_2 \), wants \( y(x_1, t) := y_1 \leq y(x_2, t) := y_2 \). Wants: \( \forall \, y < y_1, \ G(x_2, y, t) > G(x_2, y_1, t) \).

\[
G(x_2, y, t) = U_0(y) + t g\left(\frac{x_2 - y}{t}\right) = U_0(y) + t g\left(\frac{x_1 - y}{t}\right) + t g\left(\frac{x_2 - y}{t}\right) - t g\left(\frac{x_1 - y}{t}\right)
\]

\[
\geq U_0(y_1) + t g\left(\frac{x_1 - y_1}{t}\right) + t g\left(\frac{x_2 - y}{t}\right) - t g\left(\frac{x_1 - y_1}{t}\right)
\]

(wants) \( > U_0(y) + t g\left(\frac{x_2 - y_1}{t}\right) \).

So we wants

\[
g\left(\frac{x_1 - y_1}{t}\right) + g\left(\frac{x_2 - y}{t}\right) > g\left(\frac{x_1 - y}{t}\right) + g\left(\frac{x_2 - y_1}{t}\right),
\]

which can be obtained from the convexity of \( g \). \( \square \)