The characteristic: \( x = a(u)t + s \). Solution \( u(a(u)t + s, t) = h(s) \), or \( u = h(x - a(u)t) \).

We have the Rankine-Hugoniot condition: If \( u \) is a piecewise \( C^1 \) solution has discontinuity along \( y = y(t) \). Then

\[
y'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r},
\]

where

\[
u_l = \lim_{x \to y(t)^-} u(x, t), \quad u_r = \lim_{x \to y(t)^+} u(x, t).
\]

**Example:** Assume \( a'(u) > 0 \).

\[
\begin{aligned}
    u_t + (f(u))_x &= 0 \\
    u(0) &= \begin{cases} 
        u_l, & x < 0 \\
        u_r, & x > 0 
    \end{cases}
\end{aligned}
\]

Then \( y'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} := \sigma \).

If \( u_l > u_r \):

\[
    u(x, t) = \begin{cases} 
        u_l, & x < \sigma t \\
        u_r, & x > \sigma t 
    \end{cases}
\]

is the unique solution satisfies the Rankine-Hugoniot condition.

If \( u_l < u_r \), then from the characteristics, one knows that we have trouble to determine the solution \( u \) in the region between \( x = a(u_l)t \) and \( x = a(u_r)t \). One possibility is that we can define

\[
    u(x, t) = \begin{cases} 
        u_l, & x < \sigma t \\
        u_r, & x > \sigma t 
    \end{cases}
\]

for \( \sigma = \frac{f(u_l) - f(u_r)}{u_l - u_r} \). Then \( u \) is a solution satisfies the Rankine-Hugoniot condition. The problem is that this solution is not continuous, while from some physical considerations, we expect \( u \) to be continuous in this case. Note that \( u \) is constant along characteristics, we have \( u(x, t) = v(\xi \frac{x}{t}) \) for some \( v \). Then

\[
u_t + a(u)u_x = -\frac{x}{t^2} v'(\xi \frac{x}{t}) + \frac{1}{t} a(x)v'(\xi \frac{x}{t}) = 0.
\]

So

\[
a(v(\xi)) = \xi,
\]
where $\xi = \frac{x}{t}$. Then we can define a solution
\[
 u(x,t) = \begin{cases} 
 u_l, & x < a(u_l)t \\
 a^{-1}(\frac{x}{t}), & a(u_l)t < x < a(u_r)t \\
 u_r, & x > a(u_r)t. 
\end{cases}
\]

**Riemann Problem:** Initial data is piecewise constant.

Why do we care about it? The solution to Riemann problem is self similar, i.e., it satisfies $u(x,t) = \lambda^p u(\lambda x, \lambda^q t)$, for some $p, q, \forall \lambda > 0$. Self similar solution is scale invariant, which should govern the typical behavior of the solution.

**Check:** If $u = u(x,t)$ is a solution of $u_t + f(u)_x = 0$, then $u_\lambda = u(\lambda x, \lambda t)$ is also a solution. Then take $t = 0$, we have $u(\lambda x, 0) = u(x,0)$, so $u(x,0)$ must be piecewise constant (for $x > 0$, $u(x,0) \equiv u(1,0)$, for $x < 0$, $u(x) = u(-1,0)$).

Lax entropy condition (applies to $f$ convex)

The characteristic starting from either side of the discontinuity when tracked in backward time direction should hit the original time $t = 0$, i.e.,

$$a(u_l) > \sigma > a(u_r), \quad \sigma = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

**Definition 0.1.** A discontinuity is called a shock if it satisfies the Rankine-Hugoniot condition and the entropy condition.

**Definition 0.2.** We say a piecewise $C^1$ solution is admissible if all discontinuities are shock.

Consider $u_t + f(u)_x = 0$, assume that $u$ is the density.

**Idea:** Any discontinuous solution should be the (zero viscosity ) limit of the unique viscosity solution

$$u_t + f(u)_x = \epsilon u_{xx}.$$

Assume that $u(x,t)$ is an admissible travelling wave solution to $u_t + f(u)_x = 0$, then it should be the limit of a unique smooth solution of

$$u_t + f(u)_x = \epsilon u_{xx}.$$ 

**Ansatz:** $u_t + f(u)_x = \epsilon u_{xx}$.

**Travelling wave:** $u(x,t) = \phi(\frac{x-at}{\epsilon})$. Then $u_t = -\frac{a}{\epsilon} \phi'(\frac{x-at}{\epsilon})$, $f(u)_x = (f \circ \phi)'(\frac{x}{\epsilon})$.

As $\epsilon \rightarrow 0$, we expect that $\phi(\frac{x-at}{\epsilon}) \rightarrow u$, then it natural to enforce the condition $\phi(-\infty) = u_l$, $\phi(\infty) = u_r$. We can also assume that the wave is flat at $\infty$, so we assume $\phi'(-\infty) = \phi'(\infty) = 0$. Then we reduce to finding solutions to the problem

\[
\begin{cases} 
-\sigma \phi' + (f \circ \phi)' = \dot{\phi}' \\
\phi(-\infty) = u_l, \phi(\infty) = u_r \\
\phi'(-\infty) = \phi'(\infty) = 0 \end{cases}
\]

(0.3)

Then we have $-\sigma \phi + f \circ \phi + c = \phi'$, plug in $x = -\infty$, we get $-\sigma u_l + f(u_l) + c = 0$, so $c = \sigma u_l - f(u_l)$. So we consider the problem

\[
\begin{cases} 
\dot{\phi}' = -\sigma(\phi - u_l) + f(\phi) - f(u_l) \\
\phi(-\infty) = u_l, \phi(\infty) = u_r \end{cases}
\]

(0.4)
Assume \( u_l > u_r \), and assume (0.4) has a solution \( \phi \), then \( \phi' < 0 \) at at least one point.

**Claim:** \( \phi' < 0, \forall x \).

If not, there is \( x_0 \) such that \( \phi'(x_0) = 0 \). Then we consider the IVP

\[
\begin{align*}
\phi' &= -\sigma(\phi - u_l) + f(\phi) - f(u_l) \\
\phi(x_0) &= u_0
\end{align*}
\]

We note that \( \phi_1 \equiv u_0 \) is a solution of (0.5). But \( \phi \) is also a solution of (0.5), which is a contradiction. So \( \phi'(x) < 0 \) for all \( x \in \mathbb{R} \). So we have

\[-\sigma(\phi - u_l) + f(\phi) - f(u_l) < 0, \forall \phi \in (u_r, u_l).\]

**Claim 2:** \( \sigma = \frac{f(u_l) - f(u_r)}{u_l - u_r} \). This is because \( \phi'(-\infty) = \phi'(\infty) = 0 \).

So we have

\[\sigma < \frac{f(\phi) - f(u_l)}{\phi - u_l}, \forall \phi \in (u_r, u_l).\]

This is called Olenik entropy condition. Note that for Olenik entropy, we don’t require \( f \) to be convex.

If \( u_l < u_r \), we can obtain the similar result.

**Claim:** If \( u_l < u_r \), then viscosity profile \( \phi \) must satisfy \( \phi' > 0 \), and so

\[\sigma < \frac{f(\phi) - f(u_l)}{\phi - u_l}, \forall \phi \in (u_l, u_r).\]

**Definition 0.6** (Olenik entropy condition). A piecewise \( C^1 \) solution is admissible if

\[\frac{f(u_l) - f(u_r)}{u_l - u_r} < \frac{f(u) - f(u_l)}{u - u_l},\]

for all \( u \in (u_l, u_r) \) if \( u_l < u_r \), \( u \in (u_r, u_l) \) if \( u_r < u_l \).