HOMEWORK PROBLEMS SET 1

Let $U \subset \mathbb{R}^n$ be open, bounded, $n$ is the unit outward normal to $\partial U$.

1. Assume $V \subset\subset U$, and $u \in L^1(U)$. Show by example that if we have $\|D^h_i u\|_{L^1(V)} \leq C$ for all $0 < |h| < \text{dist}(V, \partial U)$, it doesn’t necessary follow that the weak derivative $\partial_{x_i} u$ exist and is in $L^1(V)$.

2. Verify that if $n > 1$, the unbounded function $u = \log \log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(U)$, for $U = B(0,1)$.

3. Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$, then $uv \in H^s(\mathbb{R}^n)$ and
\[
\|uv\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}
\]
for some constant $C$ depending only on $s$ and $n$.

4. a). A function $u \in H^2_0(U)$ is a weak solution of this boundary value problem for the biharmonic equation
\[
\begin{cases}
\Delta^2 u = f & \text{in } U \\
\partial_{\nu} u = 0 & \text{on } \partial U
\end{cases}
\]
provided
\[
\int_U \Delta u \Delta v \, dx = \int_U fv \, dx
\]
for all $v \in H^2_0(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of (0.1). (Hint: prove first that there is a constant $C > 0$, independent of $u \in H^2_0(U)$, such that $\sum_{i,j=1}^n \int_U |\partial_{x_i} \partial_{x_j} u|^2 \, dx \leq C \int_U |\Delta u|^2 \, dx$.)

b). For $f \in L^2(U)$, what can you say about the weak solutions $u \in H^2_0(U)$ of
\[
\begin{cases}
\Delta^2 u + \lambda u = f & \text{in } U \\
\partial_{\nu} u = 0 & \text{on } \partial U
\end{cases}
\]
in terms of the constant $\lambda$? What do you know about the eigenvalues of $\Delta^2$ for the given boundary condition?

5. Let $u \in H^1(U)$ have compact support and be a weak solution of the semi linear PDE
\[-\Delta u + c(u) = f, \quad \text{in } \mathbb{R}^n,
\]
where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \rightarrow \mathbb{R}$ is smooth, with $c(0) = 0$, $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$. 

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6. Explain how to define \( u \in H^1(U) \) to be a weak solution of Poisson equation with Robin boundary conditions:

\[
\begin{cases}
-\Delta u = f, & \text{in } U \\
u + \frac{\partial u}{\partial n} = 0 & \text{on } \partial U
\end{cases}
\]

Discuss the existence and uniqueness of a weak solution for a given \( f \in L^2(U) \).