1. Assume that the set of functions \( \{ f_n, f \} \) is equicontinuous and uniformly bounded on \( \Omega \subset \mathbb{R}^n \), assume further that \( f_n \rightharpoonup f \) in \( \mathcal{D}'(\Omega) \) as \( n \to \infty \). (that is, \( f_n \) converges to \( f \) in the distribution sense.) Show that \( f_n \) converges to \( f \) uniformly on compact subsets of \( \Omega \), as \( n \to \infty \).

2. Let \( k \geq 0 \), and \( f_n, f \) be distributions such that \( f_n \rightharpoonup f \) in \( \mathcal{D}'(\Omega) \) as \( n \to \infty \). Assume that \( f_n \in H^k(\Omega) \), with \( \sup_n \| f_n \|_{H^k(\Omega)} \leq M < \infty \). Show that \( f \in H^k(\Omega) \), \( \| f \|_{H^k(\Omega)} \leq M < \infty \), \( f_n \rightharpoonup f \) in \( H^k(\Omega) \).

3. Let \( H \) be a Hilbert space, \( f_n \in H, f \in H \) and \( f_n \rightharpoonup f \) in \( H \) as \( n \to \infty \). Assume further that \( \| f_n \| \to \| f \| \) as \( n \to \infty \). Show that \( \| f_n - f \| \to 0 \), as \( n \to \infty \).

4. (Gronwall) Let \( \eta, c \) be continuous functions on \( [0, T) \), satisfying
\[
\eta(t + s) - \eta(t) \leq \int_t^{t+s} c(\tau)\eta(\tau) \, d\tau, \quad \text{for all } s > 0, \ t \in [0, T - s).
\]
Show that
\[
\eta(t) \leq \eta(0)e^{\int_0^t c(s) \, ds}, \quad \text{for all } t \in [0, T).
\]

5. Let \( \epsilon > 0 \). Consider initial value problem of the viscous scalar conservation law
\[
\begin{cases}
\epsilon u_t + \partial_x f(u) = \epsilon u_{xx} \\
u(x, 0) = u_0(x) \in L^\infty(\mathbb{R})
\end{cases}
\]
(0.1)
Show that there is a unique solution \( u \in L^\infty(\mathbb{R} \times [0, \infty)) \); moreover, the solution is in \( C^\infty(\mathbb{R} \times (0, \infty)) \) by the following steps:

a). Define the map \( \tilde{u} := A[v] \) for \( v \in L^\infty(\mathbb{R} \times [0, T]) \) by
\[
\begin{cases}
\epsilon \tilde{u}_t + \partial_x f(v) = \epsilon \tilde{u}_{xx} \\
\tilde{u}(x, 0) = u_0(x)
\end{cases}
\]
(0.2)
Show that there is $T > 0$, depending only on $\|u_0\|_{L^\infty}$, such that $A$ is a contraction mapping from $L^\infty(\mathbb{R} \times [0, T]) \to L^\infty(\mathbb{R} \times [0, T])$; this implies that (0.1) has a unique solution in $L^\infty(\mathbb{R} \times [0, T])$.

b). Show that the unique solution $u \in L^\infty(\mathbb{R} \times [0, T])$ of (0.1) is in fact in $C^\infty(\mathbb{R} \times (0, T))$. (Hint: show that $A$ defines a contraction mapping on some subspace of $C^\infty(\mathbb{R} \times (0, T))$. You may want to construct this subspace by understanding $u_0 * \Phi_t$, where $\Phi_t$ is the heat kernel.)

c) Show that any solution $u \in L^\infty(\mathbb{R} \times [0, T])$ of (0.1) satisfies the maximum principle:

$$\|u(\cdot, t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$ 

6. Compute the unique entropy solution of

$$\begin{cases} 
  u_t + uu_x & = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\
  u(x, 0) & = g
\end{cases}$$

where

$$g(x) = \begin{cases} 
  1 & \text{for } x < -1 \\
  0 & \text{for } -1 < x < 0 \\
  2 & \text{for } 0 < x < 1 \\
  0 & \text{for } x > 1
\end{cases}$$

Draw a picture documenting your answer, being sure to illustrate what happens for all times $t > 0$. Show that the difference of your solution and the solution of an appropriate Riemann problem with $u_l = 1, u_r = 0$ at time $t = 0$ tends to zero as $t \to \infty$. 