ABSTRACT. Let $k$ denote a field with discrete valuation. We assume that $k$ is complete with perfect residue field $\mathfrak{f}$. Let $G$ denote the group of $k$-rational points of a reductive, linear algebraic group $G$ defined over $k$. A torus in $G$ is said to be unramified if it splits over an unramified extension of $k$. Let $\mathcal{C}$ denote the set of conjugacy classes of maximal unramified tori in $G$. Let $I^a$ denote the set of pairs $(F, T)$ where $F$ is a facet in the Bruhat-Tits building of $G$ and $T$ is a maximal $\mathfrak{f}$-anisotropic torus in $G_F$ (the connected reductive $\mathfrak{f}$-group associated to $F$). There is a natural equivalence relation, to be denoted $\sim$, on $I^a$. We show that there is a bijective correspondence between the set $I^a / \sim$ and $\mathcal{C}$.

0. INTRODUCTION

Let $k$ denote a field with discrete valuation. We assume that $k$ is complete with perfect residue field $\mathfrak{f}$. Let $G$ denote the group of $k$-rational points of a reductive, linear algebraic group $G$ defined over $k$. Let $G^0$ denote the group of $k$-rational points of the identity component $G^0$ of $G$. Let $\mathcal{B}(G)$ denote the Bruhat-Tits building of $G^0$.

A torus in $G$ is called unramified if it splits over an unramified extension of $k$. In this paper we classify the conjugacy classes of maximal unramified tori in $G$ in terms of equivalence classes of pairs $(G_F, T)$. Here $F$ is a facet in the building, $G_F$ is the connected reductive $\mathfrak{f}$-group associated to $F$, and $T$ is an $\mathfrak{f}$-anisotropic maximal torus in $G_F$.

The motivation for this result comes from harmonic analysis; specifically, from J.-L. Waldspurger’s papers [10, 11]. It is hoped that the classification scheme discussed in this paper will lead to a better understanding of the harmonic analysis problems considered in [10, 11].

We now discuss the contents of this paper.

Let $I$ denote the set of pairs $(F, T)$ where $F$ is a facet in $\mathcal{B}(G)$ and $T$ is a maximal $\mathfrak{f}$-torus in $G_F$. In §3.2 we define on $I$ an equivalence relation, denoted $\sim$.

In §3.3 we associate to each element $(F, T) \in I$ a conjugacy class $\mathcal{C}(F, T)$ of maximal unramified tori. The set $I$ is too large, so we restrict our attention to the subset $I^a$ of anisotropic pairs in $I$. A pair $(F, T) \in I$ is said to be anisotropic if the maximal $\mathfrak{f}$-split torus in $T$ coincides with the maximal $\mathfrak{f}$-split torus in the center of $G_F$. 

Date: January 26, 2001.
1991 Mathematics Subject Classification. Primary 22G25; Secondary 17B45, 20G15.
Key words and phrases. Bruhat-Tits building, maximal unramified tori, reductive group.
Supported by National Science Foundation Postdoctoral Fellowship 98-04375.
PREPRINT
We now state Theorem 3.4.1, the main result of this paper. Let $\mathcal{C}$ denote the set of conjugacy classes of maximal unramified tori in $G$.

**Theorem.** There is a bijective correspondence between $I^a / \sim$ and $\mathcal{C}$ given by the map which sends $(F, T)$ to $\mathcal{C}(F, T)$.

If our group is connected, reductive, and $k$-split, then this result can be derived from some work of Paul Gérardin [6]. If our group is connected, reductive, and unramified, then Waldspurger [10] stated a variant of this result as a hypothesis.

We remark that if $k$ is algebraically closed, then $\mathcal{C}$ and $I^a / \sim$ both have one element. In this case, the element of $\mathcal{C}$ is the conjugacy class of maximal $k$-split tori in $G$, and $I^a$ consists of those pairs $(F, T)$ where $F$ is an alcove in $B(G)$ and $T$ is a maximal torus in $G_F$.

Let $K$ denote a fixed maximal unramified extension of $k$. From Lemma 2.1.1 a maximal unramified torus in $G$ is the group of $k$-rational points of a maximal $K$-split torus which is defined over $k$. From a theorem of Steinberg, $G^o$ is quasisplit over $K$. Thus, the centralizer in $G^o$ of a maximal unramified torus is the group of $k$-rational points of a maximal $k$-torus. Since this correspondence is one-to-one, our theorem also provides a classification of the $G$-conjugacy classes of maximal $k$-tori which arise in this way.

This paper has benefitted from discussions with Jeff Adler, Roman Bezrukavnikov, Robert Kottwitz, Amritanshu Prasad, Gopal Prasad, Paul J. Sally, Jr., and Jiu-Kang Yu. It is a pleasure to thank all of these people.

1. **NOTATION**

In addition to the notation discussed in the introduction, we will require the following.

1.1. **Basic notation.** Let $k$ denote a field with discrete valuation $\nu$. We assume that $k$ is complete and the residue field $\mathfrak{f}$ is perfect.

Let $K$ be a fixed maximal unramified extension of $k$. Let $\mathfrak{F}$ denote the residue field of $K$. Note that $\mathfrak{F}$ is an algebraic closure of $\mathfrak{f}$.

Let $\Gamma = \text{Gal}(K/k)$.

Let $G$ be a linear algebraic group defined over $k$. We assume that the identity component $G^o$ of $G$ is reductive. We let $G = G(k)$, the group of $k$-rational points of $G$. Let $G^o = G^o(k)$. $\mathcal{D}G^o$ will denote the group of $k$-rational points of the derived group of $G^o$.

When we talk about a torus in $G$, we mean the group of $k$-rational points of a $k$-torus in $G^o$.

In order to avoid a proliferation of superscripts, we adopt the following convention. We shall call a subgroup of $G$ a parabolic subgroup of $G$ provided that it is a parabolic subgroup of $G^o$. We adopt a similar convention with respect to tori and Levi subgroups.

If $g, h \in G$, then $^gh = g^h g^{-1}$.

If a group $H$ acts on a set $S$, then $S^H$ denotes the set of $H$-fixed points of $S$.

1.2. **Apartments, buildings, and associated notation.** Let $B(G) = B(G, k)$ denote the (enlarged) Bruhat-Tits building of $G^o$. We identify $B(G)$ with the $\Gamma$-fixed points of $B(G, K)$, the
Bruhat-Tits building of $G^o(K)$. Let $\mathcal{B}^\text{red}(G)$ denote the reduced Bruhat-Tits building of $G^o$, that is, $\mathcal{B}^\text{red}(G) = \mathcal{B}(\mathcal{D}G^o)$.

For a $k$-Levi subgroup $M$ of $G$, we identify $\mathcal{B}(M, k)$ in $\mathcal{B}(G, k)$. There is not a canonical way to do this, but every natural embedding of $\mathcal{B}(M, k)$ in $\mathcal{B}(G, k)$ has the same image.

Given a maximal $k$-split torus $S$ defined over $k$ we have the torus $S = S(k)$ in $G$ and the corresponding apartment $A(S) = A(S, k)$ in $\mathcal{B}(G)$. Let $T$ be a maximal $K$-split $k$-torus containing $S$ [3, Corollaire 5.1.12]. We identify $A(S, k)$ with $A(T, K)^\Gamma$.

For $\Omega \subset A(S)$, we let $A(A(S), \Omega)$ denote the smallest affine subspace of $A(S)$ containing $\Omega$.

Suppose $x \in \mathcal{B}(G)$. We will denote the parahoric subgroup of $G^o$ attached to $x$ by $G_x$, and we denote its pro-unipotent radical by $G^+_x$. Note that both $G_x$ and $G^+_x$ depend only on the facet of $\mathcal{B}(G)$ to which $x$ belongs. If $F$ is a facet in $\mathcal{B}(G)$ and $x \in F$, then we define $G_F = G_x$ and $G^+_F = G^+_x$. Recall that $G_x$ is a subgroup of $\text{stab}_{G^o}(x)$. For a facet $F$ in $\mathcal{B}(G)$ the quotient $G_F / G^+_F$ is the group of $\mathfrak{f}$-rational points of a connected reductive group $G_F$ defined over $\mathfrak{f}$.

We denote the parahoric subgroup of $G^o(K)$ corresponding to $x \in \mathcal{B}(G, K)$ by $G(K)_x$. We denote the pro-unipotent radical of $G(K)_x$ by $G(K)^+_x$. The subgroups $G(K)_x$ and $G(K)^+_x$ depend only on the facet of $\mathcal{B}(G, K)$ to which $x$ belongs. If $F$ is a facet in $\mathcal{B}(G, K)$ and $x \in F$, then we define $G_K(F) = G(K)_x$ and $G(K)_F^+ = G(K)^+_x$. For a facet $F$ in $\mathcal{B}(G, K)$, the quotient $G(K)_F / G(K)_F^+$ is the group of $\mathfrak{g}$-rational points of a connected, reductive $\mathfrak{g}$-group $G_K(F)$.

Suppose $F$ is a $\Gamma$-invariant facet in $\mathcal{B}(G, K)$. In this case, $F^\Gamma = F^{\Gamma}$ is a facet in $\mathcal{B}(G)$. Moreover, we have $G_F^\Gamma = (G(K)_F)^\Gamma$, $G^+_F = (G(K)_F^+)^\Gamma$, and $G_F = G_{F^\Gamma}$ (in particular, $G_F$ is defined over $\mathfrak{f}$). Sometimes, we will abuse notation and denote by $G_F$ (resp., $G_F^+$, resp., $G(K)_F$, resp., $G(K)^+_F$) the group $G_{F^\Gamma}$ (resp., $G^+_{F^\Gamma}$, resp., $G(K)_F$, resp., $G(K)^+_F$).

2. Tori over $k$ and $\mathfrak{f}$

In this section we show how to move between tori over $\mathfrak{f}$ and tori over $k$.

2.1. Maximal unramified tori. We recall that a torus $T$ of $G$ is unramified if $T$ is the group of $k$-rational points of a $k$-torus which splits over an unramified extension of $k$. The following result will be used throughout the remainder of the paper.

**Lemma 2.1.1.** Suppose $T$ is a torus in $G$. The following statements are equivalent.

1. $T(k)$ is a maximal unramified torus of $G$.
2. $T$ is a maximal $K$-split $k$-torus of $G$.
3. $T$ is a maximal $K$-split torus of $G$ and $T$ is defined over $k$.

**Proof.** By definition, we have (1) and (2) are equivalent. Moreover, (3) implies (2).

We now show (2) implies (3). Suppose $T$ is a maximal $K$-split $k$-torus. Let $M = C_{G^o}(T)$. Then $M$ is a $K$-Levi subgroup of $G$ which is defined over $k$. Let $S'$ be a maximal $k$-split torus in $M$. From [3, Corollaire 5.1.12], there exists a maximal $K$-split torus $S \subset M$ such that $S' \subset S$ and $S$ is defined over $k$. Note that $S$ is also a maximal $K$-split torus in $G$. Since $S \subset M$, we have $T \subset S$. Since $S$ and $T$ are $K$-split $k$-tori and $T$ is maximal in $G$ with respect to this property, we must have $T = S$. \qed
2.2. From maximal unramified tori over \( k \) to tori over \( f \). Suppose \( T \) is a maximal unramified torus in \( G \). Let \( T \) denote the maximal \( K \)-split \( k \)-torus in \( G \) such that \( T = T(k) \). Define

\[
T(K)_c := \{ t \in T(K) \mid \nu(\chi(s)) = 0 \text{ for all } \chi \in X^*(T) \}
\]

From [9, §3.6.1] there is a natural embedding of \( B(T) \) in \( B(G) \); namely,

\[
B(T) = B(T, K)^G = A(T, K)^G \\
= (B(G, K)^{T(K)_c})^G = B(G, K)^{T(K)_c \cdot G}
\]

\( \subset B(G) \).

We shall always think of \( B(T) \) as being embedded in \( B(G) \) in this way. We now collect some facts about \( B(T) \).

**Lemma 2.2.1.** Suppose \( T \) is a maximal \( K \)-split torus which is defined over \( k \). Let \( T \) denote the group of \( k \)-rational points of \( T \).

1. \( B(T) \) is a nonempty, closed, convex subset of \( B(G) \). Moreover, \( B(T) \) is the union of the facets in \( B(G) \) which meet it.
2. There is a maximal \( k \)-split torus \( S \) in \( G \) such that \( B(T) \) is an affine subspace of \( A(S, k) \).
3. For all facets \( F \) in \( B(T) \), there exists \( (F, T) \in I \) such that the image of \( T(K) \cap G(K)_F \) in \( G_F(\mathfrak{g}) \) is \( T(\mathfrak{g}) \). Moreover, if \( F \) is a maximal facet in \( B(T) \), then \( (F, T) \in I^a \).
4. If \( F_1 \) and \( F_2 \) are maximal facets in \( B(T) \), then for all apartments \( A \) in \( B(G) \) containing \( F_1 \) and \( F_2 \) we have \( A(A, F_1) = A(A, F_2) \).

**Proof.** “(1)”: Since \( B(T) \) is the Bruhat-Tits building of \( T \), the first half of the statement follows from the work of Bruhat and Tits [2, 3].

For any \( \Gamma \)-invariant facet \( F \) of \( B(G, K) \), we have \( F^\Gamma = F \cap B(G) \) is a facet of \( B(G) \). Consequently, for any \( \Gamma \)-invariant facet \( F \) of \( B(T, K) \subset B(G, K) \), we have that \( F^\Gamma \) is a facet of \( B(G) \) which is contained in \( B(T) \).

“(2)”: From [1, Proposition 8.15] we can write \( T = T_s \cdot T_a \) where \( T_s \) the maximal \( k \)-split torus in \( T \) and \( T_a \) is the maximal \( k \)-anisotropic subtorus of \( T \). Let \( M = C_{G^0}(T_s) \). Then \( T \subset M \) and \( M \) is a \( k \)-Levi subgroup. Let \( M \) denote the group of \( k \)-rational points of \( M \). We have that the image of \( B(T) \) in \( B^{\text{red}}(M) \) is a point, call it \( x_T \). Let \( S \) be a maximal \( k \)-split torus in \( M \) such that the image (apartment) of \( A(S, k) \) in \( B^{\text{red}}(M) \) contains \( x_T \). Since \( T_s \subset S \), we have \( B(T) \subset A(S, k) \).

“(3)”: Suppose \( F \) is a facet in \( B(T) \). Let \( T \) be the maximal \( f \)-torus in \( G_F \) corresponding to the image of \( T(K) \cap G(K)_F \) in \( G_F(\mathfrak{g}) \). We have \( (F, T) \in I \).

Now suppose that \( F \) is a maximal facet in \( B(T) \). Choose \( T \) as in the previous paragraph. Let \( T_s \) be the maximal \( k \)-split torus in \( T \). Let \( T_s \) denote the \( f \)-split torus in \( G_F \) corresponding to the image of \( T_s(K) \cap G(K)_F \) in \( G_F(\mathfrak{g}) \). We have that \( T_s \) is the maximal \( f \)-split torus in \( T \). If we embed \( B(T_s, K) \) in \( B(T, K) \subset B(G, K) \) in the natural way, then we have \( B(T_s(k)) = B(T) \).

As in the proof of part (2) we may choose a maximal \( k \)-split torus \( S \) such that \( B(T) \subset A(S, k) \) and \( T_s \subset S \). Since \( F \) is a maximal facet in \( B(T) \), we have that an affine root of \( S \) with respect
to $G^\circ$, $k$, and $\nu$ is zero on $B(T_s(k)) = B(T)$ if and only if it is zero on $F$. It follows that $T_s$ is the maximal $\mathfrak{f}$-split torus in the center of $G_F$. Thus $(F, T) \in I^a$.

“(4)”: Let $A$ be an apartment in $B(G)$ containing $F_1$ and $F_2$. Since $F_1$ is maximal in $B(T)$ and $B(T)$ is convex, we conclude that $F_2 \subseteq A(A, F_1)$. Similarly, we have $F_1 \subseteq A(A, F_2)$. Thus $A(A, F_1) = A(A, F_2)$.

The previous lemma gives us a way to associate to a maximal unramified torus in $G$ a pair $(F, T) \in I^a$. We now examine how unique this association is.

**Lemma 2.2.2.** Suppose $T_1$ and $T_2$ are maximal $K$-split tori which are defined over $k$. If $F$ is a $\Gamma$-invariant facet in $A(T_1, K) \cap A(T_2, K)$ and the images of $T_1(K) \cap G(K)_F$ and $T_2(K) \cap G(K)_F$ in $G_F(\mathfrak{g})$ coincide, then $T_1$ and $T_2$ are $G_F^\mathfrak{g}$-conjugate.

**Proof.** Let $T$ denote the maximal torus in $G_F$ whose group of $\mathfrak{g}$-rational points is the image of $T_1(K) \cap G(K)_F$ in $G_F(\mathfrak{g})$. Note that $T$ is defined over $\mathfrak{f}$.

Let $Z$ denote the centralizer of $T_1$ in $G^\circ$. The group $Z$ is a $K$-Levi subgroup (and maximal torus) of $G$ which is defined over $k$. Note that $B(Z, K) = A(T_1, K)$ and so for all facets $F$ in $A(T_1, K)$ we have $Z(K)_F = Z(K) \cap G(K)_F$ and $Z(K)_F^\mathfrak{g} = Z(K) \cap G(K)_F^\mathfrak{g}$.

There exists an $h \in G(K)_F^\mathfrak{g}$ such that $hT_1 = T_2$. Let $\overline{h}$ denote the image of $h$ in $G_F(\mathfrak{g})$. By hypothesis, $\overline{h}T = T$. Thus, $\overline{h} \in (N_{G_F}(T_1))(\mathfrak{g})$. Consequently, there exists an $n \in (N_{G^\circ}(T_1))(K) \cap G(K)_F$ and $g \in G(K)_F^\mathfrak{g}$ such that $h = gn$. We have $T_2 = hT_1 = gT_1$.

For $\gamma \in \Gamma$, let $c_g(\gamma) := g^{-1}\gamma(g)$; $c_g$ is a one-cocycle. We will show that $c_g(\gamma) \in Z(K)_F^\mathfrak{g}$ for all $\gamma \in \Gamma$. Fix $\gamma \in \Gamma$. Since $F$ is $\Gamma$-stable and $g \in G(K)_F^\mathfrak{g}$, we have $c_g(\gamma) \in G(K)_F^\mathfrak{g}$. Since $g(\gamma)T_1 = T_1$, we have $c_g(\gamma) \in N_{G_F}(T_1)(K)$. Thus $A(T_1, K)$ is $c_g(\gamma)$-stable. If $C$ is an alcove in $A(T_1, K)$ such that $F \subset \overline{C}$, then $c_g(\gamma)$ fixes $C$ point-wise and therefore $c_g(\gamma)$ fixes $A(T_1, K)$. Thus, we conclude that $c_g(\gamma) \in Z(K)_F^\mathfrak{g}$.

Since $H^1(\Gamma, Z(K)_F^\mathfrak{g})$ is trivial, there exists $z \in Z(K)_F^\mathfrak{g}$ such that $gz$ is fixed by $\Gamma$. We have $g^zT_1 = T_2$ and $gz \in (G(K)_F^\mathfrak{g})^{-\Gamma} = G_F^\mathfrak{g}$.

### 2.3. From tori over $\mathfrak{f}$ to tori over $k$.

Suppose $(F, T) \in I$. Let $F'$ be the facet in $B(G, K)$ whose set of $\Gamma$-fixed points is $F$. In the final paragraph of the proof of [3, Proposition 5.1.10] Bruhat and Tits use [5, Exp. XI, Cor. 4.2] to show that there exists a maximal $K$-split torus $T$ such that $T$ is defined over $k$, the apartment $A(T, K)$ contains $F$, and the image of $T(K) \cap G(K)_F$ in $G_F(\mathfrak{g}) = G_F^\mathfrak{g}(\mathfrak{g})$ is $T(\mathfrak{g})$. We record this result in the following lemma.

**Lemma 2.3.1.** If $(F, T) \in I$, then there exists a maximal $K$-split torus $T$ such that $T$ is defined over $k$, the apartment $A(T, K)$ contains $F$, and the image of $T(K) \cap G(K)_F$ in $G_F(\mathfrak{g})$ is $T(\mathfrak{g})$.

### 3. The parameterization

In this section, we present a parameterization of $C$ via Bruhat-Tits theory.
3.1. **Strong associativity.** Following [7, 8], in [4, §2.3] the concept of strong associativity is developed. We recall the definition and some of its consequences.

**Definition 3.1.1.** Two facets $F_1$ and $F_2$ of $\mathcal{B}(G)$ are strongly associated if for all apartments $\mathcal{A}$ containing $F_1$ and $F_2$, we have

$$A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2).$$

**Remark 3.1.2.** Two facets $F_1, F_2$ of $\mathcal{B}(G)$ are strongly associated if and only if there exists an apartment $\mathcal{A}$ containing $F_1$ and $F_2$ such that $A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$. See [4, Lemma 2.3.3].

**Remark 3.1.3.** Suppose $F_1$ and $F_2$ are strongly associated facets in $\mathcal{B}(G)$. There is an identification of $G_{F_1}$ with $G_{F_2}$. Namely, the natural $\Gamma$-equivariant map

$$G(K)_{F_1} \cap G(K)_{F_2} \to G_{F_1}(\mathcal{F})$$

is surjective with kernel $G(K)_{F_1} \cap G(K)_{F_2} = G(K)_{F_1} \cap G(K)_{F_2}^+ = G(K)_{F_1}^{+} \cap G(K)_{F_2}^{+}$. See, for example, [4, Lemma 2.5.1].

**Definition 3.1.4.** If $F_1$ and $F_2$ are strongly associated facets in $\mathcal{B}(G)$, then we denote the natural identification of $G_{F_1}$ with $G_{F_2}$ introduced above by $G_{F_1} \cong G_{F_2}$.

3.2. **An equivalence relation on I.** We first consider the action of $G$ on $I$. Suppose $g \in G$ and $(F, T) \in I$. From Lemma 2.3.1 there exists a maximal $K$-split torus $T$ such that $T$ is defined over $k$, the apartment $\mathcal{A}(T, K)$ contains $F$, and the image of $T(K) \cap G(K)_F$ in $G_{F}(\mathcal{F})$ is $T(\mathcal{F})$.

Define

$$g(F, T) := (gF, ^gT)$$

where $^gT$ is the maximal $f$-torus in $G_{gF}$ whose group of $\mathcal{F}$-rational points coincides with the image of $^gT(K) \cap G(K)_{gF}$ in $G_{gF}(\mathcal{F})$. From Lemma 2.2.2, this definition is independent of the torus $T$ we choose to represent $T$.

We are now prepared to introduce a relation on $I$.

**Definition 3.2.1.** Suppose $(F_1, T_1)$ and $(F_2, T_2)$ are two elements of $I$. We will write $(F_1, T_1) \sim (F_2, T_2)$ provided that there exist an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ and $g \in G$ such that

1. $\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, gF_2)$ and
2. $T_1 \cong ^gT_2$ in $G_{F_1} \cong G_{gF_2}$.

**Lemma 3.2.2.** The relation $\sim$ on $I$ is an equivalence relation.

*Proof.* We will verify that the relation is transitive. The proofs that the relation is reflexive and symmetric are easier and left to the reader.

Suppose $(F_i, T_i) \in I$ for $i = 1, 2, 3$. Suppose $(F_1, T_1) \sim (F_2, T_2)$ and $(F_2, T_2) \sim (F_3, T_3)$. We want to show $(F_1, T_1) \sim (F_3, T_3)$.

There exist $g_2, g_3 \in G$ and apartments $\mathcal{A}_{12}$ and $\mathcal{A}_{23}$ in $\mathcal{B}(G)$ such that

1. $\emptyset \neq A(\mathcal{A}_{12}, F_1) = A(\mathcal{A}_{12}, g_2F_2)$
2. $\emptyset \neq A(\mathcal{A}_{23}, F_2) = A(\mathcal{A}_{23}, g_3F_3)$

and
1. $T_1 = g_2 T_2$ in $G_{F_1} = G_{g_2 F_2}$
2. $T_2 = g_3 T_3$ in $G_{F_2} = G_{g_3 F_3}$

Since $g_2 F_2 \subset A_{12} \cap g_2 A_{23}$, there exists an element $h \in G_{g_2 F_2}$ such that $h g_2 A_{23} = A_{12}$. We have

$$\emptyset \neq A(A_{12}, F_1) = A(A_{12}, g_2 A_{23}) = h g_2 A(A_{23}, F_2) = h g_2 A(A_{23}, g_3 F_3) = A(A_{12}, h g_2 g_3 F_3).$$

Moreover, we have that $G_{F_1} \cap G_{g_2 F_2} \cap G_{h g_2 g_3 F_3}$ surjects, under the natural map, onto $G_{F_1}(f)$ (resp., $G_{g_2 F_2}(f)$, resp., $G_{h g_2 g_3 F_3}(f)$). Thus, there exists $h' \in G_{F_1} \cap G_{g_2 F_2} \cap G_{h g_2 g_3 F_3}$ such that

$$T_1 = g_2 T_2 = h' g_2 T_2 = h' h g_2 g_3 T_3 \text{ in } G_{F_1} = G_{g_2 F_2} = G_{g_2 F_2} = G_{h g_2 g_3 F_3}.$$ 

3.3. A map from $I/\sim$ to $C$. From Lemmas 2.2.2 and 2.3.1, the following definition makes sense.

**Definition 3.3.1.** Suppose $(F, T) \in I$. Let $T$ be any maximal $K$-split torus such that $T$ is defined over $k$, the apartment $A(T, K)$ contains $F$, and the image of $T(K) \cap G(K)_F$ in $G_K(\mathfrak{g})$ is $T(\mathfrak{g})$. Define $C(F, T) \in C$ by setting $C(F, T)$ equal to the $G$-conjugacy class of $T(k)$.

**Remark 3.3.2.** If $g \in G$ and $(F, T) \in I$, then $C(F, T) = C(gF, gT)$.

**Lemma 3.3.3.** The map from $I$ to $C$ which sends $(F, T) \in I$ to $C(F, T)$ induces a well-defined map from $I/\sim$ to $C$.

**Proof.** Suppose $(F_1, T_1)$ and $(F_2, T_2)$ are two elements of $I$. We need to show that if $(F_1, T_1) \sim (F_2, T_2)$, then $C(F_1, T_1) = C(F_2, T_2)$.

Since $(F_1, T_1) \sim (F_2, T_2)$, there exist $g \in G$ and an apartment $A$ in $B(G)$ such that

$$\emptyset \neq A(A, F_1) = A(A, g F_2)$$

and

$$T_1 = g T_2 \text{ in } G_{F_1} = G_{g F_2}.$$ 

From Remark 3.3.2, we can assume that $g = 1$.

From Lemma 2.3.1 there exists a maximal $K$-split $k$-torus $T_2$ such that $F_2 \subset A(T_2, K)$ and the image of $T_2(K) \cap G(K)_F$ in $G_{F_2}(\mathfrak{g})$ coincides with $T_2(\mathfrak{g})$. Note that $C(F_2, T_2)$ is the $G$-conjugacy class of $T_2(k)$. It follows from Lemma 2.2.1 (2) that we can choose $h \in G_{F_2}$ such that $B(h T_2, k) \subset A$. Since $\emptyset \neq A(A, F_1) = A(A, F_2) \subset B(h T_2, k)$, we conclude that $F_1 \subset B(h T_2, k)$.

Let $T'$ denote the maximal $f$-torus in $G_{F_1}$ such that the image of $h T_2(K) \cap G(K)_{F_1}$ in $G_{F_1}(\mathfrak{g})$ coincides with $T'(\mathfrak{g})$. We have

$$T' = h T_2 \text{ in } G_{F_1} = G_{F_2}$$

and

$$T_1 = T_2 \text{ in } G_{F_1} = G_{F_2}.$$
Thus, there exists $h' \in G_{F_1} \cap G_{F_2}$ such that

$$h'T_1 = h'T_2 = h'T = T' \text{ in } G_{F_1} = G_{F_2} = G_{F_2} = G_{F_1}.$$ 

In other words, $h'T_1 = T'$ in $G_{F_1}$. We conclude from Lemma 2.2.2 that $\mathcal{C}(F_1, T_1)$ is the $G$-conjugacy class of $(h')^{-1} h T_2(k)$, i.e., $\mathcal{C}(F_1, T_1) = \mathcal{C}(F_2, T_2)$.

3.4. **A bijective correspondence.** We now prove the main result of this paper.

**Theorem 3.4.1.** There is a bijective correspondence between $I^a/\sim$ and $\mathcal{C}$ given by the map sending $(F, T)$ to $\mathcal{C}(F, T)$.

**Proof.** From Lemma 3.3.3, this map is well defined. From Lemma 2.2.1 (3) and Lemma 2.2.2 the map is surjective. It remains to show that the map is injective.

Suppose $(F_1, T_1)$ and $(F_2, T_2)$ are pairs in $I^a$ such that $\mathcal{C}(F_1, T_1) = \mathcal{C}(F_2, T_2)$. We need to show that $(F_1, T_1) \sim (F_2, T_2)$.

For $i = 1, 2$, from Lemma 2.3.1 we can choose a maximal $K$-split $k$-torus $T_i$ such that the $G$-conjugacy class of $T_i(k)$ is $\mathcal{C}(F_i, T_i)$, the apartment $A(T_i, K)$ contains $F_i$, and the image of $T_i(k) \cap G(K)_F$ in $G_{F_i}(\mathfrak{H})$ is $T_i(\mathfrak{H})$. Since $\mathcal{C}(F_1, T_1) = \mathcal{C}(F_2, T_2)$, there exists a $g \in G$ such that $^g T_2 = T_1$. Let $T = ^g T_2 = T_1$ and let $T = T(k)$.

Note that both $F_1$ and $g F_2$ lie in $A(T, K)^T = B(T)$. Since $(F_1, T_1)$ is an anisotropic pair, $F_1$ is a maximal facet in $B(T)$. Similarly, $g F_2$ is a maximal facet in $B(T)$. From Lemma 2.2.1 (4), the facets $F_1$ and $g F_2$ are strongly associated. Since the image of $T(K) \cap G(K)_{F_1} \cap G(K)_{g F_2}$ in $G_{F_1}(\mathfrak{H})$ (resp., $G_{g F_2}(\mathfrak{H})$) is $T_1(\mathfrak{H})$ (resp., $^g T_2(\mathfrak{H})$), we have

$$T_1 \overset{\sim}{\sim} T_2 \text{ in } G_{F_1} \overset{\sim}{\sim} G_{g F_2}.$$ 

**REFERENCES**


\footnote{Available at http://www.math.uchicago.edu/~debacker/links.html.


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