Homework 11

due Tuesday, December 12

(1) (20 pts) Calculate
\[ \lim_{n \to \infty} \sqrt{n^2 + n} - n. \]

(2) (20 pts) If \( \sum_{n=0}^{\infty} a_n \) converges and \( b_n \) is monotonic and bounded, show that \( \sum_{n=0}^{\infty} a_n b_n \) converges.

(3) (20 pts) For which complex numbers \( z \) does the series
\[ \sum_{n=0}^{\infty} \frac{1}{1 + z^n} \]
converge?

(4) (20 pts) Find the radius of convergence of the following power series:
(a) \( \sum_{n=0}^{\infty} n^3 z^n \)
(b) \( \sum_{n=0}^{\infty} \frac{2^n}{n^2} z^n \)
(c) \( \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n \)
(d) \( \sum_{n=0}^{\infty} \frac{\alpha^n}{3^n} z^n \)

(5) (20 pts) chapter 23, problem 18

(6) (20 pts) Suppose \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in a metric space \( X \), and that some subsequence converges. Prove that the whole sequence \( (x_n) \) also converges.

(7) (20 pts) Let \( X \) be a complete metric space with metric \( d \). A map \( f : X \to X \) is called a contraction if there exists a constant \( 0 < c < 1 \) such that for all \( x, y \in X \) we have
\[ d(f(x), f(y)) < c \, d(x, y). \]
Show that for any contraction there is \( p \in X \) such that \( f(p) = p \) (a so-called fix point).