Problem Set 3 – due October 6

See the course website for policy on collaboration.

1. We begin with a practical application of the Implicit Function Theorem! The Global Positioning System involves roughly 20 satellites which transmit regular signals down to earth. These signals travel at the speed of light, $c$. So, if a GPS receiver sitting at $(x, y, z)$ receives a signal at time $t$ which was sent from a satellite at $(x_1, y_1, z_1)$ at time $t_1$, then

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = c^2(t - t_1)^2.$$ 

The satellites have highly accurate clocks and time stamp their signals and their orbits are known with high accuracy. So the receiver can have confidence in knowing when and where a signal was sent from. However, its own clock may have drifted by some unknown amount $b$. So, if the receiver measures a signal received at time $s$, the actual time might be $s + b$.

Our receiver gets signals from four satellites, which it measures as arriving at times $s_1, s_2, s_3$ and $s_4$. The signal which arrives at $s_i$ was sent from a satellite at $(x_i, y_i, z_i)$ at time $t_i$. The receiver wants to compute $(x, y, z, b)$ as functions of $(s_1, s_2, s_3, s_4)$. The receiver also knows roughly where and when it is, so it may assume $(x, y, z, b)$ lie in a small open set in $\mathbb{R}^4$.

(a) Under what conditions will $(x, y, z, b)$ be locally given by a smooth function of $(s_1, s_2, s_3, s_4)$? You may leave your answer as a determinant without expanding it.

(b) What will $\partial x/\partial s_i$, $\partial y/\partial s_i$, $\partial z/\partial s_i$ and $\partial b/\partial s_i$ be? You may express your answer in terms of matrix operations without expanding them.

2. Show that an open subset of $\mathbb{R}^n$ is an $n$-dimensional manifold. (Hint: This is very short.)

3. Verify that the following are manifolds and give their dimensions:

(a) The set of $(x_1, y_1, x_2, y_2) \subset \mathbb{R}^4$ such that $x_1^2 + y_1^2 = 1$ and $(x_2, y_2)$ is on the line which is tangent to the unit circle at $(x_1, y_1)$.

(b) The set of $n \times n$ matrices with rank $n - 1$.

4. Let $S$ be the sphere $\{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Let $\mathbb{RP}^2$ be the image of $S$ under the map

$$f(x, y, z) = (x^2, y^2, z^2, xy, xz, yz).$$

(a) Is the map $f : S \to \mathbb{RP}^2$ injective? If not, when do we have $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$?

(b) Show that $\mathbb{RP}^2$ is a 2-dimensional manifold in $\mathbb{R}^6$.

5. Let $X \subset \mathbb{R}^n$ be a $d$-fold and let $x \in X$. Show that there is an open set $U \ni x$ such that $X \cap U$ is connected.

6. Let $V$ be a finite dimensional vector space and $X \subset V$ a $d$-fold in $V$. Define

$$TX = \{(x, \bar{v}) \in V \oplus V : x \in X, \bar{v} \in T_x X\}.$$ 

Show that $TX$ is a $2d$-fold in $V \oplus V$. 

7. Let $r$, $m$ and $n$ be positive integers, with $r \leq m$, $n$. We’ll be looking at $m \times n$ matrices divided into blocks as shown below:

$$X = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}$$

(a) Given $A$, $B$, $C$ with $A$ invertible, there is precisely one $D$ such that $X$ has rank $r$.
(b) Show that the $D$ in the previous problem is a smooth function of $A$, $B$ and $C$.
(c) Show that the set of rank $r$ matrices is a $mr + nr - r^2$ dimensional submanifold of the set of $m \times n$ matrices.