Problem Set 6 – due November 3

See course website for policy on collaboration.

1. Let $R$ be a rectangle in $\mathbb{R}^n$ and let $f$ and $g$ be bounded functions $R \to \mathbb{R}$. Prove from the definitions:
   (a) If $f(x) \leq g(x)$ for all $x \in R$, then $\int_R f \leq \int_R g$ and $\int f \leq \int g$.
   (b) We have $\int f + \int g \leq \int f + g \leq \int f + \int g$.
   (c) If $S \supset R$ is a larger rectangle, and $f : S \to \mathbb{R}$ is a bounded function with $f(x) = 0$ for $s \in S \setminus R$, then
      $\int_R f = \int_S f$ and $\int_R f = \int_S f$.

Whether or not you have proved them, you may assume the results of Question 1 in the rest of this (and all following) problem sets.

2. Let $Q$ be a closed rectangle. Let $R_1, R_2, \ldots$ be a sequence of open rectangles such that $Q \subseteq \bigcup R_i$. In this problem, we will show that $\text{Vol}(Q) \leq \sum \text{Vol}(R_i)$. For any subset $S$ of $\mathbb{R}^n$, let
   $\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$.

   (a) Show that there is a finite subset $R_{i_1}, R_{i_2}, \ldots, R_{i_N}$ of the $R_i$’s with $Q \subset \bigcup j R_{i_j}$.
   (b) Show that $\int \chi_S(x) = \int_0^1 \int_0^1 f(x,y) dx dy \leq \sum_{j=1}^N \int_0^1 \int_0^1 f(x,y) dx dy$.

   (Hint: Let $C$ be a rectangle which contains all of the $Q$ and $R_{i_j}$. Consider $\int_C \chi_Q$ and $\int_C \sum_{j=1}^N \chi_{i_j}$ and cite Question 1 liberally.)

3. The goal of this question is to construct a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that
   $\int_{x \in [0,1]} \int_{y \in [0,1]} f(x,y) < \int_{(x,y) \in [0,1] \times [0,1]} f(x,y)$.

   I think part (a) of this question is the hardest; you may want to first do parts (b) and (c).

   (a) Construct a subset $C$ of $[0,1] \times [0,1]$ such that (1) $C$ is dense in $[0,1] \times [0,1]$ but (2) for any $x \in [0,1]$, there is at most one $y$ such that $(x,y) \in C$.
   Let
   $f(x,y) = \begin{cases} 1 & (x,y) \in C \\ 0 & \text{otherwise} \end{cases}$.

   (b) Show that $\int_{x \in [0,1]} \int_{y \in [0,1]} f(x,y) = \int_{x \in [0,1]} \int_{y \in [0,1]} f(x,y) = 0$.

   (c) Show that $\int_{(x,y) \in [0,1] \times [0,1]} f(x,y) = 1$.

   The issue pointed out in this problem is an artifact of the Riemann integral; using the Lebesgue integral, if $\int_X \int_Y f(x,y)$ exists in the Lebesgue sense, then $\int_{(x,y)} f(x,y)$ exists and equals it.

4. Let $A$ be a matrix. An elementary row operation is to (1) switch two rows (2) multiply a row by a nonzero scalar or (3) add a scalar multiple of one row to another. Show that, if $A$ is an invertible matrix, then it is possible to apply elementary row operations to $A$ to turn $A$ into the identity. (This is a linear algebra lemma we will need next week.)
5. We introduce the following notation: Let $A$ be an $m \times n$ matrix, let $k \leq m$, $n$ and let $I$ be a $k$-element subset of $\{1, \ldots, n\}$ and $J$ a $k$-element subset of $\{1, 2, \ldots, n\}$. Then $A_{IJ}$ denote the matrix with rows indexed by $I$ and columns indexed by $J$. Let $A$ be a $\ell \times m$ matrix, $B$ a $m \times n$ matrix, let $k \leq \ell$, $m$ and let $L$ be a $k$-element subset of $\{1, 2, \ldots, \ell\}$ and $N$ a $k$-element subset of $\{1, 2, \ldots, n\}$. Show that

$$\det(AB)_{LN} = \sum_{M \subseteq \{1, 2, \ldots, m\}} |M| = m \det A_{LM}B_{MN}.$$ 

For clarity, we give an example:

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\det\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}\det\begin{bmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{bmatrix} + \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\det\begin{bmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}.$$ 

6. This question introduces a Lie group we will want to consider often in the future. Set

$$I = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$ 

You may assume without proof the identities:

$$I^2 = J^2 = K^2 = -\text{Id}_4, \quad IJ = -JI = K, \quad IK = -KI = -J, \quad JK = -KJ = I.$$ 

Let $\mathbb{H} = \text{Span}_\mathbb{R}(\text{Id}_4, I, J, K) \subset \text{Mat}_{4\times 4}(\mathbb{R})$. As an abstract ring, $\mathbb{H}$ is called the quaternions. For a quaternion $\alpha = a\text{Id}_4 + bI + cJ + dK$, we define $\overline{\alpha} = a\text{Id}_4 - bI - cJ - dK$.

(a) Check that $\overline{\alpha\beta} = \overline{\beta} \cdot \overline{\alpha}$. (The $\cdot$ on the right hand side is multiplication.)

(b) Define $SU(2) = \{\alpha \in \mathbb{H} : \alpha\overline{\alpha} = 1\}$. Show that $SU(2)$ is a subgroup of $\text{GL}_4$.

(c) Define $\mathfrak{su}(2) = \text{Span}_\mathbb{R}(I, J, K)$. Show that $\mathfrak{su}(2)$ is the Lie algebra of $SU(2)$.

(d) Let $X = pI + qJ + rK \in \mathfrak{su}(2)$ and define $\theta^2 = p^2 + q^2 + r^2$. Show that

$$\exp(X) = \cos \theta \text{Id}_4 + \frac{\sin \theta}{\theta} X.$$ 

(e) Show that $\exp$ is injective on $\{pI + qJ + rK : p^2 + q^2 + r^2 < \pi^2\}$ and describe how $\exp$ behaves on the sphere $\{pI + qJ + rK : p^2 + q^2 + r^2 = \pi^2\}$.