Problem Set 7 – due November 10

See course website for policy on collaboration.

1. (a) Let \( c(r, \theta, h) = (r \cos \theta, r \sin \theta, h) \).

Compute \( \det Dc \) and describe when it is zero.

(b) Let \( s(r, \theta, \phi) = (r \cos \phi \cos \theta, r \cos \phi \sin \theta, r \sin \phi) \).

Compute \( \det Ds \) and describe when it is zero.

2. In this question, we will prove that rotating coordinates in \( \mathbb{R}^2 \) does not change integrals. Do not simply quote the change of variables formula, as the whole point of this proof is to walk you through a special case which is missing a lot of the complications.

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function and \( R > 0 \) a constant such that \( f(x, y) = 0 \) for \( (x, y) \not\in [-R, R]^2 \).

Let’s assume that \( f \) is bounded and is continuous except on a set of measure 0, so \( \int_{\mathbb{R}^2} f \) exists.

(Since \( f = 0 \) for \( (x, y) \) large, we just have \( \int_{\mathbb{R}^2} f = \int_{[-R, R] \times [-R, R]} f \).

(a) Let \( h \in \mathbb{R} \). Show that
\[
\int_{\mathbb{R}^2} f(x, y) = \int_{\mathbb{R}^2} f(x, y + hx).
\]

(b) Let \( a \) and \( b > 0 \). Show that
\[
\int_{\mathbb{R}^2} f(ax, by) = \frac{1}{ab} \int_{\mathbb{R}^2} f(x, y).
\]

(c) Let \( \theta \) be a real number. Show that the map \( (x, y) \mapsto (\cos \theta x - \sin \theta y, \sin \theta x + \cos \theta y) \) can be written as a composition of maps of the forms \( (x, y) \mapsto (x, y + hx) \), \( (x, y) \mapsto (x + hy, y) \) and \( (x, y) \mapsto (ax, by) \).

(d) Show that
\[
\int_{\mathbb{R}^2} f(x, y) = \int_{\mathbb{R}^2} f(\cos \theta x - \sin \theta y, \sin \theta x + \cos \theta y).
\]

3. This problem provides a quick proof that \( \lim_{R \to \infty} \int_{[0, R]} \frac{\sin x}{x} \) exists (where \( x = 0 \) may be filled in however you like.)

(a) Integrate by parts to show, for \( 0 < M < N \), that
\[
\left| \int_{[M, N]} \frac{\sin x}{x} \right| \leq \frac{3}{M}.
\]

(If you get a different constant, that’s fine.)

(b) Show that \( \lim_{R \to \infty} \int_{[0, R]} \frac{\sin x}{x} \) exists.

4. In order to make proofs in this problem shorter, we will slightly broaden the definition of measure zero: We’ll say that \( X \subseteq \mathbb{R}^n \) has measure zero if, for any \( \epsilon > 0 \), there exist a sequence of closed bounded sets \( R_i \) such that \( \text{Vol}(R_i) \) exists, \( X \subseteq \bigcup R_i \) and \( \sum \text{Vol}(R_i) < \epsilon \). (In class, we insisted that the \( R_i \) be rectangles.)

Moreover, if a high school geometry student would know how to compute \( \text{Vol}(R_i) \), you may assert its value without proof.
(a) Let $ f : [0, 1] \to \mathbb{R}^2 $ be a smooth function. Show that there is a constant $ C > 0 $ such that, for any $ 0 \leq x < y \leq 1 $, the image $ f([x, y]) $ is contained in a ball of radius $ C(y - x) $.

(b) Show that $ f([0, 1]) $ has measure 0.

Let $ g : [0, 1] \times [0, 1] \to \mathbb{R}^2 $ be smooth. Let $ Z $ be the set of $ (x, y) \in [0, 1] \times [0, 1] $ where $ \det(Dg)_{(x,y)} = 0 $. The goal of the next several parts is to show that $ g(Z) $ has measure 0.

(c) Let $ z \in Z $. Show that there is a nonzero vector $ \vec{v} \in \mathbb{R}^2 $ such that

$$ \frac{\partial}{\partial x} (\vec{v} \cdot g) = \frac{\partial}{\partial y} (\vec{v} \cdot g) = 0 \text{ at } z $$

(d) Show that there is a constant $ C > 0 $ such that, for any $ z \in Z $ and any $ \epsilon \times \epsilon $ rectangle $ r $ containing $ z $, the image $ g(r) $ is contained in a shape of area $ < C\epsilon^3 $. Here “shape” can mean anything whose area a high school geometry student could compute.

(e) Show that $ g(X) $ has measure 0.