PROBLEM SET 8 – due November 17

See course website for policy on collaboration.
This week, we compute integrals! We also prove a useful formula for the matrix exponential.

1. In this problem, we will compute \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

(a) Give exhaustions of \( \mathbb{R}^2 \) and \( \mathbb{R} \) to show that
\[
\int_{\mathbb{R}^2} e^{-x^2-y^2} = \left( \int_{\mathbb{R}} e^{-x^2} \right)^2.
\]

(b) Give another exhaustion of \( \mathbb{R}^2 \) to show that
\[
\int_{\mathbb{R}^2} e^{-x^2-y^2} = \lim_{R \to \infty} \int_{x^2+y^2 \leq R^2} e^{-x^2-y^2}.
\]

(c) Compute \( \int_{x^2+y^2 \leq R^2} e^{-x^2-y^2} \). Hint: Try the change of variables \( x = r \cos \theta, \, y = r \sin \theta \).

2. (a) Show that, for \( 0 < r < 1 \), we have
\[
\int_{(x,y) \in [0,r] \times [0,r]} \frac{1}{1-xy} = \sum_{n=1}^{\infty} \frac{r^{2n}}{n^2}.
\]

(b) Show that
\[
\int_{(x,y) \in [0,1] \times [0,1]} \frac{1}{1-xy} = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

(c) Make the substitution \( x = u + v, \, y = u - v \) to compute that \( \int_{(x,y) \in [0,1] \times [0,1]} \frac{1}{1-xy} = \frac{\pi^2}{6} \).

3. In this problem, we will compute \( \lim_{M \to 0} \int_{[0,M]} \frac{\sin x}{x} \).

(a) Show that
\[
\lim_{M \to \infty} \lim_{N \to \infty} \int_{[0,M] \times [0,N]} e^{-xy} \sin x = \lim_{M \to 0} \int_{[0,M]} \frac{\sin x}{x}.
\]

(b) Show that
\[
\lim_{N \to \infty} \lim_{M \to \infty} \int_{[0,M] \times [0,N]} e^{-xy} \sin x = \lim_{N \to \infty} \int_{[0,N]} \frac{1}{1+y^2} = \frac{\pi}{2}.
\]

We now need to justify interchanging the limits. Feel free to replace the bounds below with other bounds which do the job.

(c) Show that
\[
\left| \int_{[0,M_1] \times [N_1,N_2]} e^{-xy} \sin x \right| \leq \frac{1}{N_1}.
\]

Hint: \( |\sin x| \leq x \).

(d) Show that
\[
\left| \int_{[M_1,M_2] \times [0,N_1]} e^{-xy} \sin x \right| \leq \frac{3}{M_1} + \frac{1}{N_1}.
\]

Hint: I’d integrate on \( y \) first.

(e) Show that
\[
\lim_{M \to \infty} \lim_{N \to \infty} \int_{[0,M] \times [0,N]} e^{-xy} \sin x = \lim_{N \to \infty} \lim_{M \to \infty} \int_{[0,M] \times [0,N]} e^{-xy} \sin x.
\]
4. Let \( X \) and \( Y \) be \( n \times n \) matrices. Recall that we write \( \text{ad}_X \) for the map \( Y \mapsto [X,Y] \). The point of this problem is to prove the identity:

\[
(D \exp)_X(Y) \ e^{-X} = \sum_{n=0}^{\infty} \frac{\text{ad}^n_X(Y)}{(n+1)!}.
\]

I have to admit I don’t know a really nice proof of this one; here is an argument.

(a) Check that

\[
\frac{d}{ds} e^{sX} = X e^{sX} = e^{sX} X.
\]

(b) Put

\[
U(s) = \left. \frac{\partial}{\partial t} e^{s(X+ty)} \right|_{t=0} e^{-sX}.
\]

Check that

\[
\frac{dU}{ds} = e^{sX} Y e^{-sX}.
\]

(c) Show that

\[
U(s) = \sum_{n=0}^{\infty} \frac{\text{ad}^n_X(Y)s^{n+1}}{(n+1)!}
\]

and deduce the identity for \((D \exp)_X(Y)\).