Problem Set 8 – due November 5

See the course website for policy on collaboration.

Definitions/Notation Some more definitions regarding Grassmannians: If V has a chosen basis $e_1, \ldots, e_n$ and we have $v_1 \wedge \cdots \wedge v_d = \sum_{1 \leq i_1 < \cdots < i_d \leq n} p_{i_1 \cdots i_d} e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_d}$, then we call the $(d)$ numbers $p_{i_1 \cdots i_d}$ the **Plücker coordinates** of Span$_k (v_1, \ldots, v_d)$. If we choose a decomposition of $V$ as $A \oplus B$, where dim $A = d$, then get embedding $s : \text{Hom}(A, B) \hookrightarrow G(d, V)$, so that $L(s(f))$ is the graph of $f$. This is a dense open subset of $G(d, V)$, isomorphic to $A^d (\text{dim } V - d)$. We’ll call this a **Schubert chart** (this is not standard terminology).

If $I$ is a graded radical ideal in $k[x_1, \ldots, x_n]$, with corresponding projective variety $X \subset \mathbb{P}^{n-1}$, we define the **Hilbert function** of $X$ by $h_X^{\text{func}}(t) = \dim_k k[x_1, \ldots, x_n]/I_t$. We define the Hilbert polynomial $h_X^{\text{poly}}(t)$ to be the unique polynomial such that $h_X^{\text{poly}}(t) = h_X^{\text{func}}(t)$ for $t \gg 0$. We showed that $h(t)$ has degree $X$ and defined the degree of $X$ is the integer deg $X$ such that the leading term of $h(t)$ is $\frac{\deg X}{(\text{dim } X)!} t^{\deg X}$.

**Problem 1** Let $W$ be the 2-dimensional subspace of $k^5$ spanned by the rows of

\[
\begin{pmatrix}
1 & 0 & 2 & 3 & 5 \\
0 & 1 & 7 & 11 & 13
\end{pmatrix}
\]

(a) Compute the Plücker coordinates of $W$. (That is to say, the $\binom{5}{2}$ numbers $p_{12}, p_{13}, p_{14}, p_{15}, p_{23}, p_{24}, p_{25}, p_{34}, p_{35}, p_{45}$.)

(b) Let $W^\perp$ be $\{ \vec{x} : \vec{w} \cdot \vec{x} = 0 \text{ for all } \vec{w} \in W \}$. (Here $\cdot$ is the dot product.) Find a basis of $W^\perp$.

(c) Compute the $\binom{5}{3}$ Plücker coordinates of $W^\perp$: (That is to say, the $\binom{5}{3}$ numbers $p_{123}, p_{124}, p_{125}, p_{134}, p_{135}, p_{145}, p_{234}, p_{235}, p_{245}, p_{345}$.)

**Problem 2.** (a) For $W \subset k^n$, let $W^\perp$ be $\{ \vec{x} : \vec{w} \cdot \vec{x} = 0 \text{ for all } \vec{w} \in W \}$. (Again, $\cdot$ is the dot product.) Show that the correspondence $W \mapsto W^\perp$ is a regular map $G(d, k^n) \to G(n - d, k^n)$.

(b) Find a formula expressing the Plücker coordinates of $W^\perp$ in terms of the Plücker coordinates of $W$. (The formula I am thinking of is extremely explicit and fairly simple.)

**Problem 3** Let $f(w, x, y, z)$ and $g(w, x, y, z)$ be relatively prime nonzero homogenous polynomials of degrees $a$ and $b$. Compute the Hilbert polynomial of $Z(f, g)$ in $\mathbb{P}^3$.

**Problem 4** Let $X$ be $\mathbb{P}^1 \times \mathbb{P}^2$ embedded into $\mathbb{P}^5$ by the Segre map $(p : q) \times (r : s : t) \mapsto (pr : ps : pt : qr : qs : qt)$. Compute the Hilbert polynomial and degree of $X$ in $\mathbb{P}^5$. Hint: It is easier and more useful to describe the quotient $k[x_1, x_2, x_3, x_4, x_5, x_6]/I$ than to give generators for $I$.

**Problem 5** Let $C$ be a conic in $\mathbb{P}^2$ and let $p_1, p_2, p_3, p_4, p_5$ and $p_6$ be 6 distinct points on $C$. We write $L_{ij}$ for the line through $p_i$ and $p_j$, and write $\lambda_{ij}$ for a linear equation defining $L_{ij}$. In this problem we will use Bezout’s theorem to prove:

**Pascal’s Theorem:** The three points $L_{12} \cap L_{45}, L_{23} \cap L_{56}$ and $L_{34} \cap L_{16}$ are colinear.

(a) Show there is a constant $c \in k$ such that $\lambda_{12}\lambda_{34}\lambda_{56} + c\lambda_{23}\lambda_{34}\lambda_{56}$ vanishes at 7 points of $C$.

(b) Define $E = Z(\lambda_{12}\lambda_{34}\lambda_{56} + c\lambda_{23}\lambda_{34}\lambda_{56})$. Show that $E = C \cup M$ for some line $M$.

(c) Show that $M$ passes through the points $L_{12} \cap L_{45}, L_{23} \cap L_{56}$ and $L_{34} \cap L_{16}$.

This was a short problem set. How about taking some of the extra time to read/think for your final paper?