Problem Set 9 – due December 9

See the course website for policy on collaboration.

Problem 1. (a) Write \( d(x^2 + y^2) \) in terms of \( dx \) and \( dy \).
(b) Let \( g(u, v) = (e^u \cos v, e^u \sin v) \). Compute \( g^* d(x^2 + y^2) \) directly from your answer to (a).
(c) Check that the result is equal to \( d\left((e^x \cos y)^2 + (e^x \sin y)^2\right) \).

Problem 2 Let \( \gamma \) be a circle in \( \mathbb{R}^2 \) centered at 0, oriented counter clockwise. Compute \( \int_{\gamma} xdy - ydx \).

Problem 3 This problem asks you to verify by brute force something which we will soon have much better tools to prove.
Let \( A \) and \( B \) be open sets in \( \mathbb{R}^2 \). We write \((u,v)\) for the coordinates on \( A \) and \((x,y)\) for the coordinates on \( B \). Let \( \phi \) be a smooth map from \( A \) to \( B \). Let \( \omega = p(x,y)dx + q(x,y)dy \) be a differential form on \( B \) and let \( \phi^*\omega = f(u,v)du + g(u,v)dv \).
Show that \( \frac{\partial g}{\partial u} - \frac{\partial f}{\partial v} = \det(D\phi) \left( \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) \).

Problem 4 Let
\[
\mathcal{Q} = \left\{ (\alpha, \beta) : \begin{array}{c}
0 < \alpha, \beta \\
2\alpha + \beta, \alpha + 2\beta < \pi 
\end{array} \right\}
\]
\[
\mathcal{A} = \left\{ (x,y) : e^{-x} + e^{-y} < 1 \right\}
\]
Consider the system of equations
\[
e^{-x} \sin \alpha = e^{-y} \sin \beta \quad (\star)
e^{-x} \cos \alpha + e^{-y} \cos \beta = 1
\]
(a) Show that there is a bijection \((x,y) \to (\alpha(x,y), \beta(x,y))\) from \( \mathcal{Q} \) to \( \mathcal{A} \) so that \((x,y,\alpha(x,y),\beta(x,y))\) obeys the equations \((\star)\). We’ll use this to think of \( \alpha \) and \( \beta \) as functions on \( \mathcal{Q} \).
(b) Write down two (nonzero, non-proportional) linear relations between the four 1-forms \( dx, dy, d\alpha \) and \( d\beta \) on \( \mathcal{Q} \).
(c) Show that
\[
det \begin{pmatrix}
\frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\
\frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y}
\end{pmatrix} = 1.
\]
Explain why this implies that \( \mathcal{A} \) and \( \mathcal{Q} \) have the same area. (The slickest way to do this is to write the Jacobian matrix as the product of two simpler \( 2 \times 2 \) matrices, so that you never need to compute the individual entries of the Jacobian.)
(d) Compute the area of \( \mathcal{Q} \).
(e) Show that the area of \( \mathcal{T} \) is
\[
\int_0^{\infty} -\log(1 - e^{-x})dx.
\]
(f) Show that the integral in (e) is equal to \( \sum_{n=1}^{\infty} \frac{1}{n^2} \). (Yes, you need to justify moving the sum past the integral.)