Problem Set 5
Due on October 25

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Please also submit a starred problem from this, or a previous problem set.

Problem 1. Let $a, b \in \mathbb{R}_{>0}$. Prove that the matrix

$$
\begin{pmatrix}
\cosh(\sqrt{ab}t) & \sqrt{ab} \sinh(\sqrt{ab}t) \\
\sqrt{bt/a} \sinh(\sqrt{ab}t) & \cosh(\sqrt{ab}t)
\end{pmatrix}
$$

lies in $GL_2(\mathbb{R}(t))_{\geq 0}$. (Hint: this matrix is an analogue of the exponentials $e^{\gamma t}$ in the factorization of TP functions.)

Problem 2. Suppose $A(t) \in GL_n(\mathbb{R}(t))$. Show that the rows of the matrix $X_A$, considered as vectors in $\mathbb{R}^\infty$, are linearly independent.

Problem 3. Let $N$ be a cylindric network (acyclic, directed from one boundary to the other, weighted). Let $X = M(N)$ (the path generating function matrix of $N$) have folding $A(t)$. Show (using network combinatorics) that the $k \times k$ minors of $A(t)$ have nonnegative coefficients when $k$ is odd, and have sign-alternating coefficients when $k$ is even. For example, when $k = 1$, the claim is that all the coefficients of $a_{ij}(t)$ are nonnegative.

Problem 4. Fix $n > 1$.

(1) What is the determinant of the (folded versions of) curl matrices $N(a_1, \ldots, a_n)$ and whirl matrices $M(a_1, \ldots, a_n)$?

(2) Let $A(t) \in \tilde{U}_{\geq 0} \subset GL_n(\mathbb{R}(t))_{\geq 0}$. Suppose in the factorization theorem we have

$$
A(t) = \left( \prod_{i=1}^\infty N(a^{(i)}) \right) Y(t) \left( \prod_{i=-\infty}^{-1} M(a^{(i)}) \right).
$$

What is the relationship between $\det A(t)$ and $\det Y(t)$?

(3) Suppose $n = 2$. Prove that $\det Y(t)$ in the previous theorem is equal to $e^{\gamma t}$. (Hint: let $B(t) = A(t)^{-1}$. Then compare the Edrei-Thoma factorization of $b_{11}(t)$ with that of $a_{22}(t)$.)

(4) Now allow $n$ to be arbitrary. As we remarked in class, each entry $a_{ij}(t)$ is a TP function and thus,

$$
a_{ij}(t) = e^{\gamma t} \prod_{i=1}^\infty \frac{(1 + \beta_i t)}{(1 - \alpha_i t)}.
$$

Prove that if $n > 1$ then $\gamma = 0$. (Hint: first reduce to $n = 2$. Then use the previous problem.)

(5) Prove that the function $\det(Y(t))$ from (2) is equal to the function 1.

(6) (*) (Open?) Let $A(t) \in \tilde{U}_{\geq 0}$. Then each $a_{ij}(t)$ is a TP function with no “exponential part”. What is the relationship between the poles and zeroes of different $a_{ij}(t)$?

Problem 5. (*) This problem is about the generators $x_i(a) \in U_{\geq 0}$. Recall (and check if you never did!) the relation:

$$
x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}(bc/(a + c))x_i(a + c)x_{i+1}(ab/(a + c)).
$$
As shorthand we may write the above relation as \((i, i + 1, i) \leftrightarrow (i + 1, i, i + 1)\). There is also a commutation relation corresponding to \((i, j) \leftrightarrow (j, i)\) for \(|i - j| > 1\).

(1) Due to laziness, commas are omitted in the following. Prove that the sequence of relations

\[
(121321) \leftrightarrow (212321) \leftrightarrow (213231) \leftrightarrow (231213) \leftrightarrow \\
(232123) \leftrightarrow (323123) \leftrightarrow (321323) \leftrightarrow (312323) \leftrightarrow (321232) \leftrightarrow (312132) \leftrightarrow (132312) \\
(123212) \leftrightarrow (123121) \leftrightarrow (121321)
\]

changes the parameters \(a, b, c, d, e, f\) in \(x_1(a)x_2(b)x_1(c)x_3(d)x_2(e)x_1(f)\) back to themselves. (Hint: given what we have proved, this is supposed to be very easy.)

(2) Two reduced words for \(w \in S_n\) are commutation equivalent if they are related by the moves \(ij \sim ji\) for \(|i - j| > 1\). Let \(G_w\) be the graph on commutation equivalent classes of reduced words of \(w\), with edges whenever two (representatives of commutation classes of) reduced words are related by a move \(i(i + 1)i \sim (i + 1)i(i + 1)\).

Prove that the fundamental group of \(G_w\) is generated by 4-cycles of the form

\[
i, i + 1, i \cdots j, j + 1, j \sim i + 1, i, i + 1 \cdots j, j + 1, j \\
i + 1, i, i + 1 \cdots j + 1, j, j + 1 \sim i, i + 1, i, j + 1, j, j + 1 \sim i, i + 1, i \cdots j, j + 1, j
\]

and an 8-cycle corresponding to the sequence of moves in the previous problem. (The sequence in the previous part looks longer than an 8-cycle, but after one removes the commutation moves, it becomes an 8-cycle).

**Problem 6. (\*) (Open?)** Let \(X = \prod_{i=-\infty}^{\infty} N(a^{(i)})\) be a bi-infinite product of curls such that the sum of all the parameters is bounded. Factor \(X\) into canonical form \(X = AYB\) where \(A\) is a product of curls, \(B\) is a product of whirls, and \(Y\) is doubly entire.